Imperial College London

Brane Solution and Dimension Reduction in Supergravity

Author: Jiakang Chen

Supervisor: Prof. K.S. Stelle

Assessor: Prof. D. Waldram

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Abstract

This dissertation delves into the intricate structures and dynamics of supergravity, focusing on 11-dimensional supergravity and its extensions through consistent truncation and dimensional reduction. The exploration begins with the application of the p-brane ansatz to derive distinct brane solutions, such as 2-branes and 5-branes, and extends to the study of multi-brane configurations to understand the orbital dynamics of probe branes. A detailed investigation of the dimensional reduction process, particularly the S^1 reduction, systematically reduces the dimensional setting and examines the implications for the supergravity action. This approach is expanded to successive reductions represented as T^n, analysing modifications in the Lagrangian and the emergence of scalar symmetries alongside their associated coset spaces. The dissertation also addresses T-duality within supergravity, focusing on resolving the chirality issues in type IIA and IIB theories by reducing them to nine dimensions, which facilitates their unification and the application of T-duality. The insights gained from this study link to broader theoretical frameworks like M-theory, and future directions in the field are discussed. This work significantly enhances our understanding of the landscape of string theory and supergravity, highlighting new possibilities for theoretical advances and practical applications.

Contents

Chapter 1

Introduction

There are four fundamental forces in the universe: electromagnetism, the strong force, the weak force, and gravity. Unifying these forces has become a major goal for theoretical physics. After centuries of work, the Standard Model was constructed to describe the unified nongravitational fields. This leaves gravity, and one of the central problems in modern physics is to unify it with the other forces. This endeavour is known as quantum gravity.

Supergravity is among the most well-known topics in quantum gravity. It is a gravitational theory that combines general relativity and supersymmetry. Although it encounters problems in treating ultraviolet divergences [1], its mathematical structure under certain constraints is still remarkable. In this theory, we discuss dynamics based on a set of new objects called "branes". We can calculate their equations of motion under certain spacetimes.

We can relate zero-dimensional branes, or 0-branes, to black holes. This allows the dynamics of black holes to be studied in the framework of supergravity [2]. The analogies between 0-brane solutions and Schwarzschild black holes help us to gain a better understanding of their behaviour near the event horizon. Another important example of a brane is the 1-brane, which represents a string in supergravity. This builds a connection from supergravity to superstring theory. With duality symmetry and dimensional reduction, one can convert 11-dimensional supergravity into 10-dimensional superstring theory. For higher dimensions, we have D-branes, which can describe the interaction between strings [3]. This link leads to a unified approach called M-theory [4], which includes type I string theory, type IIA string theory, type IIB string theory, heterotic $E_8 \times E_8$ string theory, heterotic SO(32) string theory and 11D supergravity.

The method of dimensional reduction in supergravity is called Kaluza–Klein dimensional reduction. It provides an effective approach to constructing solutions for 11D supergravity. By leveraging solutions in lower dimensions, one can gain insights into the higher-dimensional problem [5]. Two major approaches are diagonal and vertical dimensional reduction [6]. However, dimensional reduction causes the number of scalars to increase. For example, we obtain an extra scalar in 10 dimensions. Accumulation of scalars can make calculations more complicated. To deal with this, we use duality symmetry.

Instead of studying M-theory directly, we start with a basic discussion of brane motion and possible solutions. This will assist in our understanding of supergravity and M-theory. In Chapter 2, we will present some basic theoretical frameworks that support our discussion. In Chapter 3, we explore the realm of 11-dimensional supergravity and its extension to general D-dimensional frameworks via consistent truncation. Our journey begins with the application of the p-brane ansatz to derive specific brane solutions. The choice of ansatz will lead to two distinct types of solutions. To validate these findings, we will examine the most straightforward scenario of $D = 11$, which will demonstrate the emergence of 2-brane and 5-brane solutions. The chapter concludes with an investigation into multi-brane configurations, examining the orbital dynamics of a probe brane influenced by a much larger background brane.

Chapter 4 focuses on the crucial technique of dimensional reduction, which was briefly introduced in Chapter 3. We start by detailing the S^1 reduction process, which systematically reduces the setting from $(D+1)$ to D dimensions [7], and we discuss the implications of this truncation for the supergravity action. This concept is further extended to include successive reductions, represented as T^n. Throughout the chapter, we will explore the modifications to the Lagrangian at each dimensional level and conclude with an examination of the scalar symmetries that emerge during reduction, particularly their coset spaces.

In Chapter 5, we discuss T-duality in the framework of supergravity, in which we cannot link type IIA and IIB theories in 10 dimensions because of their inconsistent chirality. Our exploration will centre on the application of dimensional reduction techniques to type IIA and IIB superstring theories, reducing them to nine dimensions. This allows the chiral property of the type IIB theory to be removed, which leads to the unification of these two theories.

In the final chapter of this dissertation, we will review the main discoveries and ideas discussed in the previous chapters. Each part of this work has deepened our understanding of complex theories, such as string theory and supergravity. We will summarise these insights and show how they connect to M-theory. Additionally, we will look ahead at possible future developments in this field.

Chapter 2

Theoretical Background

A suitable theoretical framework and definitions are necessary for the derivation of brane solutions and subsequent discussion. In this chapter, we will introduce several new concepts, including forms, and operations on them.

2.1 Forms and the Wedge Product

An n-form is a completely antisymmetric tensor of type (0, n), represented as

$$V_{\mu_1 \dots \mu_n} = V_{[\mu_1 \dots \mu_n]} = V_{[n]}. \tag{2.1}$$

As a cotensor, V can be expanded using the tensor product of differential forms:

$$V = V_{i_1 \dots i_n} dx^{i_1} \otimes \dots \otimes dx^{i_n}. \tag{2.2}$$

Transitioning to an n-form basis, $dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_n}$, yields

$$V = \frac{1}{n!} V_{i_1 \dots i_n} dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_n}, \tag{2.3}$$

where $\wedge$ denotes the wedge product, which combines differential forms. Given a p-form A and a q-form B, their wedge product, a $(p+q)$-form, is given as follows:

$$(A \wedge B)_{\mu_1 \cdots \mu_{p+q}} = \frac{(p+q)!}{p!q!} A_{[\mu_1 \cdots \mu_p} B_{\mu_{p+1} \cdots \mu_{p+q}]}. \tag{2.4}$$

For example, the wedge product of two one-forms is

$$(A \wedge B)_{\mu\nu} = 2A_{[\mu} B_{\nu]} = A_\mu B_\nu - A_\nu B_\mu. \tag{2.5}$$

Additionally, the wedge product satisfies the following antisymmetry property:

$$A \wedge B = (-1)^{pq} B \wedge A. \tag{2.6}$$

2.2 Exterior Derivative

We define an exterior derivative d as an operator that takes an n-form V to an $(n+1)$-form dV:

$$(dV)_{i_1 \dots i_{n+1}} = (n+1) \partial_{[i_1} V_{i_2 \dots i_{n+1}]}. \tag{2.7}$$

One familiar example is the Maxwell field strength F, which is the exterior derivative of the potential A:

$$F = dA = 2\partial_{[\mu}A_{\nu]}, \tag{2.8}$$

where we use a one-form potential. If we have a wedge product of a p-form ω and a q-form η, the exterior derivative obeys the Leibniz rule:

$$\mathrm{d}(\omega \wedge \eta) = (\mathrm{d}\omega) \wedge \eta + (-1)^p \omega \wedge (\mathrm{d}\eta), \tag{2.9}$$

which results in a $(p+q-1)$-form.

2.3 Levi–Civita Tensor

The Levi–Civita symbol is defined as follows:

$$\tilde{\epsilon}_{\mu_1\mu_2\cdots\mu_n} = \begin{cases} +1 & \text{if } \mu_1\mu_2\cdots\mu_n \text{ is an even permutation of } 01\cdots(n-1), \\ -1 & \text{if } \mu_1\mu_2\cdots\mu_n \text{ is an odd permutation of } 01\cdots(n-1), \\ 0 & \text{otherwise,} \end{cases} \tag{2.10}$$

which is completely antisymmetric. We then define the Levi–Civita tensor as follows:

$$\epsilon_{\mu_1\mu_2\cdots\mu_n} = \sqrt{|g|}\tilde{\epsilon}_{\mu_1\mu_2\cdots\mu_n}, \tag{2.11}$$

where g is the determinant of the metric $g_{\mu\nu}$. The contraction rule of Levi–Civita tensors is

$$\epsilon^{\mu_1\mu_2\cdots\mu_p\alpha_1\cdots\alpha_{n-p}}\epsilon_{\mu_1\mu_2\cdots\mu_p\beta_1\cdots\beta_{n-p}} = (-1)^s p!(n-p)!\delta^{[\alpha_1}_{\beta_1}\cdots\delta^{\alpha_{n-p}]}_{\beta_{n-p}}, \tag{2.12}$$

where $s = 1$ is the number of time coordinates on the manifold.

2.4 Hodge Duality

We define the Hodge star operator on an n-dimensional manifold as a map from p-forms to $(n-p)$-forms:

$$(*A)_{\mu_1\cdots\mu_{n-p}} = \frac{1}{p!}\epsilon^{v_1\cdots v_p}{}_{\mu_1\cdots\mu_{n-p}}A_{v_1\cdots v_p}. \tag{2.13}$$

Applying the Hodge star twice yields either the original form or its negation:

$$**A = (-1)^{s+p(n-p)}A. \tag{2.14}$$

One can therefore find that using the Hodge star and the exterior derivative gives the following divergence:

$$(*d*A)_{\mu_1\ldots\mu_{p-1}} = (-1)^{s+p(n-p)}\nabla^\nu A_{\mu_1\ldots\mu_{p-1}\nu}. \tag{2.15}$$

2.5 Integration

Using (2.10) and (2.11), one can define a volume n-form:

$$\epsilon = \sqrt{|g|}dx^1 \wedge \ldots \wedge dx^n = \frac{1}{n!}\sqrt{|g|}\tilde{\epsilon}_{\mu_1\ldots\mu_n}dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_n}, \tag{2.16}$$

where the volume element $d^n x$ can be turned into a tensor density form such as that in (2.3):

$$d^n x = \mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^{n-1}. \tag{2.17}$$

Using (2.16) and (2.17), we can write an n-form tensor into

$$\begin{aligned}
\epsilon &\equiv \epsilon_{\mu_1 \cdots \mu_n}\ \mathrm{d}x^{\mu_1} \otimes \cdots \otimes \mathrm{d}x^{\mu_n} \\
&= \frac{1}{n!} \epsilon_{\mu_1 \cdots \mu_n}\ \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_n} \\
&= \frac{1}{n!} \sqrt{|g|} \tilde{\epsilon}_{\mu_1 \cdots \mu_n}\ \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_n} \\
&= \sqrt{|g|} \mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^{n-1} \\
&\equiv \sqrt{|g|} d^n x.
\end{aligned} \tag{2.18}$$

Combining the ideas we have introduced throughout this section, the Hodge duality of a 0-form can be defined as follows:

$$*1 = \epsilon, \tag{2.19}$$

The integral I of a scalar function ϕ over an n-manifold is written as follows:

$$I = \int \phi(x) \sqrt{|g|} d^n x = \int \phi(x) * 1. \tag{2.20}$$

Finally, if we have two p-forms A and B, the simple form can be obtained as follows:

$$*A \wedge B = \frac{1}{p!} A_{\mu_1 \cdots \mu_p} B^{\mu_1 \cdots \mu_p} * 1. \tag{2.21}$$

Chapter 3

Brane Solution in Supergravity

In this chapter, we will explore the brane solution within the framework of supergravity, beginning our discussion with an 11-dimensional bosonic action. This foundational action enables us to derive the equations of motion (EoMs) for each component. Following this, we apply a consistent truncation to the action, reducing it to a system in D dimensions. This process allows us to obtain specific EoMs related to curvature and field strength.

By employing an ansatz on the metric, we can solve the EoMs, yielding a detailed solution in a generalised D-dimensional form. Upon obtaining the brane solution, we specifically set $D = 11$ to examine the properties of the brane within the supergravity framework. Additionally, we derive the motion of the brane under certain assumptions, providing deeper insights into its behaviour.

3.1 11D Supergravity

It is crucial to first elucidate why the selection of 11 dimensions significantly simplifies our equations. This particular choice is intimately linked to the unique properties of supersymmetry algebra, as highlighted in the following equation [8]:

$$\{Q, Q\} = C\left(\Gamma^A P_A + \Gamma^{AB} U_{AB} + \Gamma^{ABCDE} V_{ABCDE}\right), \tag{3.1}$$

where U and V are charges derived later in the discussion. Here, C represents the charge conjugation matrix and P the energy–momentum tensor. This algebraic structure ensures that, in our supergravity framework, we only need to consider the metric g_{MN}, the gauge field $A_{[n]}$ and the gravitino ϕ. However, this form does not exist in lower-dimensional supergravity.

Upon reducing the dimensionality, additional scalar fields, known as dilatons, emerge. Conversely, in dimensions exceeding 11, a supergravity theory does not exist; this limitation is dictated by supersymmetry itself, which prohibits spins higher than 2, and inherently caps the maximum number of supercharges at 32, aligning precisely with the component structure of 11-dimensional supergravity.

Let us start with the bosonic action [9] with a vanishing fermionic gravitino:[1]

$$I_{11} = \int d^{11}x \left\{\sqrt{-g}\left(R - \frac{1}{48}F_{[4]}^2\right)\right\} - \frac{1}{6}\int F_{[4]} \wedge F_{[4]} \wedge A_{[3]}. \tag{3.2}$$

[1]In maximal supergravity, we usually assume a spin 3/2 gravitino.

Here we use g to represent the determinant of the metric and $F_{[4]}$ for the field strength generated by a 3-form antisymmetric gauge potential. To find the EoM for the gauge potential of (3.2), we apply the variation δI_{11}. Using the properties of forms given by (2.2), (2.13) and (2.16), one can find that[2]

$$\delta(F_{[4]}^2) \propto \delta\left(F_{[4]} \wedge *F_{[4]}\right) \propto \delta A_{[3]} \wedge d * F_{[4]}. \tag{3.3}$$

Then, the EoM with respect to $A_{[3]}$ is as follows:

$$d * F_{[4]} + \frac{1}{2} F_{[4]} \wedge F_{[4]} = 0. \tag{3.4}$$

Hence, one can find the conserved quantity from the above equation. Considering the order of the form, this quantity can be constructed as an integral over a 7-form space with an 8-dimensional boundary ∂M_8 [10]:

$$U = \int_{\partial \mathcal{M}_8} \left({}^* F_{[4]} + \frac{1}{2} A_{[3]} \wedge F_{[4]} \right), \tag{3.5}$$

which is termed "electric" charge. We can also find the Bianchi identity $dF_{[4]} = 0$, which gives rise to another conserved quantity:

$$V = \int_{\partial \widetilde{\mathcal{M}}_5} F_{[4]}. \tag{3.6}$$

This is termed "magnetic" charge and it has a 5-dimensional boundary $\partial \tilde{M}_5$. Both U and V contribute to the supersymmetry algebra described by (3.1). To discuss this charge, we first need to consider its supergravity solutions, the simplest of which is found by applying the "p-brane ansatz". Further details are provided in the next section.

3.2 Single-Charge Action

The structure of 11-dimensional supergravity is clear and elegant, but it needs to connect to a quantum theory of gravity. It would be convenient for us to consider a general form that contains metric, scalar and gauge potentials. This is known as an effective field theory and couples supergravity with matter fields. A similar configuration, called the Neveu–Schwarz/Neveu–Schwarz (NS-NS) sector, is achieved in superstring theory. For example, one can have an NS-NS sector under a string theory framework [11] as follows:

$$\begin{aligned} I_{\text{eff}} = \textstyle\int d^D x \sqrt{-g} e^{-2\phi} \big[(D-26) - \tfrac{3}{2}\alpha' \big(R + 4\nabla^2\phi - 4(\nabla\phi)^2 \\ - \tfrac{1}{12} F_{MNP} F^{MNP} + \mathcal{O}\left(\alpha'\right)^2 \big] . \end{aligned} \tag{3.7}$$

Our target action for supergravity is similar and needs to satisfy the dimension limit. In string theory, we have a maximum of 26 dimensions, which can be seen in the term $(D-26)$. Analogously, we replace it with $(D-10)$ since string theory in the supergravity framework has a maximum of 10 dimensions. Using a suitable Weyl transformation[3] and

[2] The detailed derivation is trivial, but it is necessary to be careful with the coefficient. The full calculation can be found in Section A.1.

[3] We shall discuss a systematic method to obtain a reduced string theory in Chapter 4.

frame changing, one can obtain a specialised version of the effective action in $D = 10$ [8]:

$$\begin{aligned} I_{\text{IIA}}^{\text{string}} = \int d^{10}x\sqrt{-g^{(\text{s})}} \Big\{ e^{-2\phi} \Big[R\left(g^{(\text{s})}\right) + 4\nabla_M\phi\nabla^M\phi - \frac{1}{12}F_{MNP}F^{MNP} \Big] \\ -\frac{1}{48}F_{MNPQ}F^{MNPQ} - \frac{1}{4}\mathcal{F}_{MN}\mathcal{F}^{MN} \Big\} + \mathcal{L}_{FFA}, \end{aligned} \tag{3.8}$$

where the last term is called the Chern–Simons term.

In (3.8), we have seen a combination of different field strengths, scalars and metrics. For explanatory purposes, we apply a consistent truncation [12] to this effective action that constrains the equation to have only one field strength, one scalar and one metric:

$$I = \int d^Dx\sqrt{-g}\left[R - \frac{1}{2}\nabla_M\phi\nabla^M\phi - \frac{1}{2n!}e^{a\phi}F_n^2 \right]. \tag{3.9}$$

This is known as a single-charge action. Since the solutions obtained from the truncated theory can be applied to the untruncated theory [8], we will later set $D = 11$ to find the original solution.

Varying (3.9) with respect to g_{MN}, $A_{[n-1]}$ and ϕ, we can obtain a set of EoMs:[4]

$$\begin{aligned} &R_{MN} = \frac{1}{2}\partial_M\phi\partial_N\phi + S_{MN}, \\ &S_{MN} = \frac{1}{2(n-1)!}e^{a\phi}\left(F_M \ldots F_N \cdots - \frac{n-1}{n(D-2)}F^2 g_{MN} \right), \\ &\nabla_{M_1}\left(e^{a\phi}F^{M_1\cdots M_n}\right) = 0, \\ &\Box\phi = \frac{a}{2n!}e^{a\phi}F^2. \end{aligned} \tag{3.10}$$

To solve all the EoMs, it is necessary to make a different ansatz. Let us consider the metric first. Our solutions need to preserve certain unbroken supersymmetries and translational symmetries. These requirements can be fulfilled in our construction of the ansatz as it has $(\text{Poincaré})_d \times \text{SO}(D-d)$ symmetry. Hence, we consider a total D-dimensional space that is composed of a d-dimensional hyperplane, also known as a worldvolume, and a $(D-d)$-dimensional transverse space. Subsequently, the spacetime coordinates can be split into $x^M = (x^\mu, x^m)$. Variables with Greek letter indices represent coordinates on the worldvolume and satisfy the $(\text{Poincaré})_d$ isometries. To distinguish the two ranges, we use variables with Latin letter indices to represent the transverse space coordinates, which follow the $\text{SO}(D-d)$ isometries. Accordingly, one can find the metric

$$ds^2 = e^{2A(r)}dx^\mu dx^\nu\eta_{\mu\nu} + e^{2B(r)}dy^m dy^n\delta_{mn}, \tag{3.11}$$

where $\mu(\nu) = 0, 1, \cdots, d-1$ and $m(n) = d, \cdots, D-1$. Both functions $A(r)$ and $B(r)$ depend on $r = \sqrt{y^m y^m}$, which is the isotropic radial coordinate. Let us also set the scalar field to depend only on r. Once we have the ansatz for the metric, we can then try to derive its corresponding curvature.

3.3 Construction of Curvature

To solve the EoMs in (3.10), we need to find the form of the curvature R_{MN}. In this section, we will use the metric ansatz to calculate its corresponding curvature in both

[4] The full calculations for EoMs can be found in Section A.2.

worldvolume and transverse space. For a flat space, we can go through the simple path in general relativity, that is, from $g_{\mu\nu}$ to the Christoffel symbol and then to the Ricci tensor. However, our situation is more complicated because we consider not only two ranges, but also the non-flat metric. We have two different ways to tackle this complex metric. The first goes from a vielbein to a spin connection, which is equivalent to the curvature term. The second uses a classic approach to construct the curvature, but with Weyl transformations. Both methods are useful and provide insights for deeper discussions. We will therefore derive our results using both methods.

Let us first consider using a vielbein. This is a set of orthogonal metric tensors that couple the spacetime metric of a curved manifold with a flat Minkowsi metric of a tangent space:

$$g_{MN} = e_M{}^{\underline{E}} e_N{}^{\underline{F}} \eta_{\underline{EF}}, \tag{3.12}$$

where the capital Latin letter index with underline $\underline{E}(\underline{F})$ is the tangent–space index. By substituting our ansatz, the vielbein 1-forms can obtain the following:

$$e^{\underline{\mu}} = e^{A(r)} dx^\mu, \qquad e^{\underline{m}} = e^{B(r)} dy^m. \tag{3.13}$$

Here we split the tangent–space index $\underline{E} = (\underline{\mu}, \underline{m})$ similarly to the worldvolume spacetime.

In the Cartan formalism [13], the vielbein we defined above can also be regarded as a spin connection. One can construct the torsion and curvature via these spin connections [14] as follows:

$$\begin{aligned} \Theta^{\underline{E}} &= \mathrm{d}e^{\underline{E}} + \omega^{\underline{E}}{}_{\underline{F}} \wedge e^{\underline{F}}, \\ R^{\underline{EF}} &= d\omega^{\underline{EF}} + \omega^{\underline{ED}} \wedge \omega_{\underline{D}}{}^{\underline{F}}. \end{aligned} \tag{3.14}$$

These are known as Cartan's structure equations. We can then find the curvature with given $\omega^{\underline{EF}}$. In our discussion, the simplest manifolds are considered, which means that the related torsion is zero.

By taking $\underline{E}(\underline{F}) = \underline{\mu}, \underline{m}$, we can obtain three different combinations with indices $\underline{\mu\nu}$, $\underline{\mu n}$ and $\underline{mn}$. The substitution[5] of the vielbein leads to the following expression:

$$\begin{aligned} \omega^{\underline{\mu v}} &= 0, \\ \omega^{\underline{\mu m}} &= e^{-B(r)} \partial_m A(r) e^{\underline{\mu}}, \\ \omega^{\underline{mn}} &= e^{-B(r)} \partial_n B(r) e^{\underline{m}} - e^{-B(r)} \partial_m B(r) e^{\underline{n}}. \end{aligned} \tag{3.15}$$

Before plugging the 1-form spin connections into the second Cartan structure equation, it is convenient to raise the index $\underline{D}$ via $\eta_{\underline{IJ}}$ to obtain

$$R^{\underline{EF}} = d\omega^{\underline{EF}} + \eta_{\underline{IJ}} \omega^{\underline{EI}} \wedge \omega^{\underline{JF}}. \tag{3.16}$$

Upon obtaining the full expression of the curvature in the tangent space, the next step is to convert the underlined index to our original index. This can be achieved by

$$R_{MN} = R_{\underline{MN}} e^{\underline{M}}{}_M e^{\underline{N}}{}_N, \tag{3.17}$$

where we have $e^{\underline{\mu}}{}_\nu = \delta^\mu{}_\nu e^{A(r)}$ and $e^{\underline{m}}{}_n = \delta^\mu{}_\nu e^{B(r)}$, and the remaining combinations vanish.

[5] More details are presented in Section A.3.

Finally, we can express the Ricci tensor in the spacetime of our supergravity theory as[6]

$$
\begin{aligned}
R_{\mu\nu} =& - \eta_{\mu\nu} e^{2(A-B)} \left(A'' + dA'^2 + \tilde{d}A'B' + \frac{\tilde{d}+1}{r} A' \right), \\
R_{mn} =& - \delta_{mn} \left(B'' + dA'B' + \tilde{d}B'^2 + \frac{2\tilde{d}+1}{r} B' + \frac{d}{r} A' \right) \\
& - \frac{y^m y^n}{r^2} \left(\tilde{d}B'' + dA'' - 2dA'B' + dA'^2 - \tilde{d}B'^2 - \frac{\tilde{d}}{r} B' - \frac{d}{r} A' \right),
\end{aligned}
\tag{3.18}
$$

where we use the chain rule[7] to vary the derivative term of the functions A and B. Thus, the prime denotes the derivative with respect to the radius r. We also have $\tilde{d} = D - d - 2$, which represents the dimension of the dual space.

Now, we use a Weyl transformation to construct the curvature. Instead of using the previous condition, it is more intuitive to start with a general situation. Let us assume $\bar{g}_{\mu\nu}$ is a general Lorentzian metric and $\tilde{g}_{ij}$ a general Riemannian metric. These two metrics are initially flat; in other words, their curvatures are zero before applying the Weyl transformation. Beginning with a general $g_{\mu\nu}$, we can perform a Weyl transformation $\hat{g}_{\mu\nu} = \Lambda^2 g_{\mu\nu}$. The corresponding change in the Christoffel symbol is as follows:

$$
\begin{aligned}
\hat{\Gamma}^{\mu}_{\nu\rho} &= \frac{1}{2} \hat{g}^{\mu\sigma} \left(\hat{g}_{\sigma v,\rho} + \hat{g}_{\sigma\rho,v} - \hat{g}_{v\rho,\sigma} \right) \\
&= \frac{1}{2} \Lambda^{-2} g^{\mu\sigma} \left(\partial_\rho \left(\Lambda^2 g_{\sigma v} \right) + \partial_v \left(\Lambda^2 g_{\sigma\rho} \right) - \partial_\sigma \left(\Lambda^2 g_{v\rho} \right) \right) \\
&= \Gamma^{\mu}_{v\rho} + \Lambda^{-1} g^{\mu\sigma} \left(\partial_\rho \Lambda g_{\sigma v} + \partial_\nu \Lambda g_{\sigma\rho} - \partial_\sigma \Lambda g_{v\rho} \right).
\end{aligned}
\tag{3.19}
$$

For simplicity, we label the extracted part as $S^{\mu}_{\nu\rho}$. Hence we can derive the transformed Riemann tensor as follows:

$$
\hat{R}^{\mu}{}_{\nu\rho\sigma} = R^{\mu}{}_{\nu\rho\sigma} + \left(\nabla_\rho S^{\mu}_{\sigma\nu} + S^{\mu}_{\delta\rho} S^{\delta}_{\sigma\nu} - \rho \leftrightarrow \sigma \right). \tag{3.20}
$$

We can further determine the transformed Ricci tensor for the single-metric case. In our situation, it is necessary to consider two ranges of indices. Fortunately, we can regard the two metrics as block diagonal form metrics that only depend on their own coordinates. In this case, we can use the above transforms. Furthermore, because we have two different factors in terms of the functions A and B, a single Weyl transformation is insufficient to obtain the ansatz used in (3.11). Thus, we need to perform a two-step transformation. First, we set $\Lambda = e^{B-A}$, which will only influence the transverse space as the initial value when the worldvolume curvature is zero. Subsequently, we perform the second transformation with $\Lambda = e^{A}$, which affects the total space. This will remove the $A(r)$ dependence in transverse space and ensure that the final form of our metric is the same as (3.11). Finally, we obtain the following Weyl-transformed curvature:

$$
\begin{aligned}
R_{\nu\sigma} =& \bar{R}_{\nu\sigma} - e^{2(A-B)} \left(\widetilde{\nabla}^2 A + \widetilde{g}^{ij} \partial_i A \left(d\partial_j A + \tilde{d}\partial_j B \right) \right) \bar{g}_{\nu\sigma}, \\
R_{ij} =& \widetilde{R}_{ij} - d\widetilde{\nabla}_i \widetilde{\nabla}_j A - \tilde{d}\widetilde{\nabla}_i \widetilde{\nabla}_j B + \tilde{d}\partial_i B \partial_j B - d\partial_i A \partial_j A \\
& + 2d\partial_{(i} A \partial_{j)} B - \left(\widetilde{\nabla}^2 B + \widetilde{g}^{ij} \partial_i B \left(d\partial_j A + \tilde{d}\partial_j B \right) \right) \widetilde{g}_{ij},
\end{aligned}
\tag{3.21}
$$

[6] We present these transformations step by step in Section A.4.

[7] We follow these relations: $\partial_m A = A' r^{-1} y^m$ and $\partial_n \partial_m A = A'' r^{-2} y^m y^n - A' r^{-3} y^m y^n + A' r^{-1} \delta_{mn}$.

where d and $\tilde{d}$ have the same definitions as previously. Here, $\bar{R}_{\nu\sigma}$ and $\widetilde{R}_{ij}$ are the original curvatures before transformation and their corresponding covariant derivatives are $\bar{\nabla}$ and $\widetilde{\nabla}$. As the functions A and B do not have any dependence on worldvolume coordinates, only derivatives with respect to transverse coordinates are preserved. Considering the same chain rule as before, one can find an equality between (3.18) and (3.21). Both methods give the same results.

In this section, we have successfully determined the curvature from our metric ansatz. However, we have not considered the ansatz for potential because there are different ways to form a field strength satisfying our assumptions. We will discuss all possible ansatzes in the next subsection.

3.4 Electric and Magnetic Branes

We have two possible ansatzes, related by duality. Usually, these are termed the "electric" and "magnetic" ansatzes. In this section, we will discuss them separately. We can then apply all ansatzes back to the EoMs to find the values of e^A and e^B.

3.4.1 Electric Brane Ansatz

Let us consider the first possibility. To find an n-form field strength $F_{[n]}$, one can start with an $(n-1)$-form gauge potential $A_{[n-1]}$. This choice is similar to the Maxwell gauge potential, which always couples to a charged particle, and this is why we call it the electric ansatz. It can be expressed as

$$A_{\mu_1\cdots\mu_{n-1}} = \epsilon_{\mu_1\cdots\mu_{n-1}} e^{C(r)}, \tag{3.22}$$

which couples to an $(n-2)$-dimensional charged object. Our worldvolume dimension can be shown to be $d_{el} = n-1$. The corresponding field strength is

$$F^{(\mathrm{el})}_{m\mu_1\cdots\mu_{n-1}} = \epsilon_{\mu_1\cdots\mu_{n-1}} \partial_m e^{C(r)}. \tag{3.23}$$

Because $C(r)$ depends only on the radius of the transverse space, the field strength has one transverse index m and preserves the required symmetries. Now, we have all ansatzes and are ready to solve (3.10). Plugging the curvature and field strength into the EoMs, a set of equations can be found, leading to the final solutions for the metric and scalar field:

$$\begin{aligned}
A'' + d\,(A')^2 + \tilde{d}A'B' + \frac{(\tilde{d}+1)}{r}A' &= \frac{\tilde{d}}{2(D-2)}S^2, \\
B'' + dA'B' + \tilde{d}\,(B')^2 + \frac{(2\tilde{d}+1)}{r}B' + \frac{d}{r}A' &= -\frac{d}{2(D-2)}S^2, \\
\tilde{d}B'' + dA'' - 2dA'B' + d\,(A')^2 - \tilde{d}\,(B')^2 & \\
-\frac{\tilde{d}}{r}B' - \frac{d}{r}A' + \frac{1}{2}(\phi')^2 &= \frac{1}{2}S^2, \\
\phi'' + dA'\phi' + \tilde{d}B'\phi' + \frac{(\tilde{d}+1)}{r}\phi' &= -\frac{1}{2}aS^2.
\end{aligned} \tag{3.24}$$

Here, we define S in terms of our gauge potential ansatz:

$$S = C' e^{\frac{1}{2}a\phi - dA + C}. \tag{3.25}$$

Although we have simplified a large part of the EoMs, the results are still complicated. To further reduce the equations, a linear condition is applied:

$$dA' + \tilde{d}B' = 0, \tag{3.26}$$

due to the supersymmetry. In [15], a detailed proof is presented for this linear requirement. Since we remove the dependence on the function B through the above condition, a form of isotropic Laplacian is observed in the simplified EoMs:[8]

$$\nabla^2\phi = \frac{D-1}{r}\phi' + \phi''. \tag{3.27}$$

When we rewrite the EoMs in Laplacian form, the scalar ϕ and the function A are surprisingly linked by the defined term S. Brief algebraic manipulation shows that the first two equations in (3.24) can be combined to remove the dependence on A:

$$\phi' = \frac{-a(D-2)}{\tilde{d}}A'. \tag{3.28}$$

At the final stage, we have the last EoM that only depends on ϕ. Again, this can be expressed as a Laplace equation:

$$\nabla^2\phi + \frac{\Delta}{2a}(\phi')^2 = 0 \Rightarrow \nabla^2 e^{\frac{\Delta}{2a}\phi} = 0, \tag{3.29}$$

where Δ is a new combination of constants and dimensions.[9] Thus, one can assume the solution for our scalar field ϕ is also the solution of a Laplace equation:

$$H(y) = e^{\frac{\Delta}{2a}\phi}. \tag{3.30}$$

It is simple to write ϕ in terms of $H(y)$ and therefore as $A(r)/B(r)$. Then, the final solution based on the electric ansatz can be written as follows:

$$\begin{aligned} ds^2 &= H^{\frac{-4\tilde{d}}{\Delta(D-2)}} dx^\mu dx^\nu \eta_{\mu\nu} + H^{\frac{4d}{\Delta(D-2)}} dy^m dy^m, \\ A_{\mu_1\cdots\mu_{n-1}} &= \frac{2}{\sqrt{\Delta}}\epsilon_{\mu_1\cdots\mu_{n-1}} H(y)^{-1}. \end{aligned} \tag{3.31}$$

Let us consider the specific form for our $H(y)$. Since it satisfies the Laplace equation, it must preserve the isotropic symmetry. In that case, the simplest assumption that can be used is

$$H(y) = 1 + \frac{k}{r^{\tilde{d}}}, \tag{3.32}$$

where we ensure the positivity of the constant k due to the restriction of singularity at finite r.

Once we find the general D-dimensional solution, the corresponding expression at maximal supergravity can be derived by setting $D = 11$. It is important to restate the difference between the 11-dimensional case and lower dimensions, which indicate that there is no scalar field in a maximum supergravity theory. However, our general solution

[8] Here we have a general expression for the isotropic Laplacian. In our calculation, we replace $(D-1)$ with $(\tilde{d}-1)$.

[9] The expression for Δ is $\frac{2d\tilde{d}}{(D-2)} + a^2$.

is the truncated theory that contains scalar fields. To fix this problem, one can set all coefficients in front of the scalar ϕ to be zero, that is, $a = 0$. By applying this restriction, the value of Δ can be found to be 4 in the $D = 11$ case. As we started with an electric ansatz, we can obtain an $A_{[3]}$ gauge field and an $F_{[4]}$ antisymmetric field strength in the $D = 11$ case. Then, the worldvolume dimension is $d = (n-1) = 3$ and the corresponding dual dimension is $\tilde{d} = 6$. Using (3.32) and (3.31), one can show that the electric ansatz solution is as follows:

$$\begin{aligned} ds^2 &= \left(1 + \frac{k}{r^6}\right)^{-2/3} dx^\mu dx^\nu \eta_{\mu\nu} + \left(1 + \frac{k}{r^6}\right)^{1/3} dy^m dy^m, \\ A_{\mu\nu\lambda} &= \epsilon_{\mu\nu\lambda} \left(1 + \frac{k}{r^6}\right)^{-1}, \end{aligned} \tag{3.33}$$

which we also term the membrane solution or the 2-brane solution as $p = d - 1 = 2$ in this case.

3.4.2 Magnetic Brane Ansatz

As mentioned previously, another possible ansatz exists for the antisymmetric field strength. One can start with a dualised field strength, which can be related to the original $F_{[n]}$. According to the definition of the Hodge dual, we can express this as $*F$, which is a $(D-n)$ form. Then, the dualised gauge field has a $(D-n-1)$ form and its worldvolume dimension can be defined as $d = (D - n - 1)$. To compare different ansatzes, it is convenient to write the magnetic ansatz with $F_{[n]}$ rather than its dual. This field strength only has a transverse direction [8]:[10]

$$F_{m_1 \cdots m_n} = \lambda \epsilon_{m_1 \cdots m_n p} \frac{y^p}{r^{n+1}}. \tag{3.34}$$

We find that there is a coefficient λ, unlike in the electric case. This is called the magnetic-charge parameter, which will be determined later as all assumptions are applied.

Since we only have a different choice of field strength and the remaining ansatzes stay the same, it is not surprising that we obtain the same form of solution as before:

$$\begin{aligned} ds^2 &= H^{\frac{-4\tilde{d}'}{\Delta(D-2)}} dx^\mu dx^\nu \eta_{\mu\nu} + H^{\frac{4d'}{\Delta(D-2)}} dy^m dy^m, \\ F_{m_1 \cdots m_n} &= -\frac{2}{\sqrt{\Delta}} \epsilon_{m_1 \cdots m_n r} \partial_r H(y), \quad H(y) = e^{-\frac{\Delta}{2a}\phi}, \end{aligned} \tag{3.35}$$

where we label the worldvolume dimension as d' to distinguish it from the electric case. We also have a minus sign in front of ϕ in the structure of $H(y)$, which comes from a different S value[11] in (3.24). Recalling that we have the specific form of $H(y)$, we can further derive the relationship between k and λ:

$$k = \frac{\sqrt{\Delta}}{2\tilde{d}} \lambda. \tag{3.36}$$

[10] The form of the field strength is guaranteed by the Bianchi identity, which is examined in Section A.5.

[11] In the magnetic case, $S = \lambda \left(e^{\frac{1}{2}a\phi - \tilde{d}B}\right) r^{-\tilde{d}-1}$.

In $D = 11$ supergravity, we have a 6-dimensional worldvolume, and thus a 5-brane for the magnetic ansatz. The solution can be written as

$$\begin{aligned} ds^2 &= \left(1+\frac{k}{r^3}\right)^{-1/3} dx^\mu dx^\nu \eta_{\mu\nu} + \left(1+\frac{k}{r^3}\right)^{2/3} dy^m dy^m, \\ F_{m_1\cdots m_4} &= 3k\epsilon_{m_1\cdots m_4 p}\frac{y^p}{r^5}. \end{aligned} \tag{3.37}$$

To summarise, we have applied two different antisymmetric tensor ansatzes to obtain an electric brane and a magnetic brane. Then, we used the general p-brane solution and set $D = 11$ with a vanishing scalar field, which leads to an M2-brane and an M5-brane. In other scenarios, additional types of branes are possible; however, these depend on the total spacetime dimensions. Detailed exploration of these variants is beyond the scope of this discussion.

3.5 Brane Motion

In the previous subsection, we discussed a p-brane ansatz that considers a single-charge system. Now, we discuss multi-charge situations. One physical approach to multi-charge solutions is known as brane coupling. Specifically, we can put a light brane as a probe into a background with a heavy brane. By using a suitable probe action, we can then investigate the motion of our probe under certain background conditions.

Generally, one can start with a p-brane probe with a supergravity background, and can express the probe action as [8], [16]

$$I_{\text{probe}} = -T_\alpha \int d^{p+1}\xi \left(-\det\left(\partial_\mu x^m \partial_\nu x^n g_{mn}(x)\right)\right)^{\frac{1}{2}} e^{-\frac{1}{2}\varsigma^{\text{pr}}\vec{a}_\alpha\cdot\vec{\phi}} + Q_\alpha \int \tilde{A}^\alpha_{[p+1]}. \tag{3.38}$$

Here, $\varsigma^{\text{pr}} = \pm 1$ and $\tilde{A}^\alpha_{[p+1]}$ is the gauge potential:

$$\tilde{A}^\alpha_{[p+1]} = [(p+1)!]^{-1}\partial_{\mu_1}x^{m_1}\cdots\partial_{\mu_{p+1}}x^{m_{p+1}}A^\alpha_{m_1\cdots m_{p+1}}d\xi^{\mu_1}\wedge\cdots\wedge d\xi^{\mu_{p+1}}. \tag{3.39}$$

In our discussion, we choose the relatively simple case of a $D = 11$ membrane coupling to a heavy membrane background. In that case, we have $\vec{a}_\alpha \cdot \vec{\phi} = 0$, and let ξ^μ be x^μ. Then, the general probe action (3.38) becomes

$$I_{\text{probe}} = -T \int d^3x \left(\sqrt{-\det\left(e^{2A(y)}\eta_{\mu\nu}\partial_0 x^0\partial_0 x^0 + e^{2B(y)}\partial_\mu y^m \partial_\nu y^m\right)} - m\partial_0 x^0 e^{C(y)}\right), \tag{3.40}$$

where we can relabel $\partial_0 x^0$ as $\dot{t}$. To solve this action, we need to deal with the square root term first. A simple expansion is not an ideal solution, so we introduce a method to add a new variable called an einbein.

3.5.1 Einbeins

Let us consider an example to show how einbeins work and that is independent of our final results. Considering a particle moving from A to B, its corresponding action can be written as follows:

$$I\left[x^\mu(\lambda)\right] = -m\int_{\lambda_A}^{\lambda_B} d\lambda \left[-g_{\mu\nu}(x(\lambda))\dot{x}^\mu\dot{x}^\nu\right]^{1/2}, \tag{3.41}$$

where we have a square root term. Now, we introduce an einbein e:

$$\hat{I}\left[x^{\mu}(\lambda), e(\lambda)\right] = \frac{1}{2}\int_{\lambda_A}^{\lambda_B} d\lambda \left[e^{-1} g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} - m^2 e\right], \tag{3.42}$$

which helps us to eliminate the square root term. However, we cannot arbitrarily add new variables into our EoM without affecting the invariance of the system. In other words, one needs to prove that I and $\hat{I}$ will give the same EoMs. Therefore, variations with respect to x^{μ} and e must be zero, which gives us the following expression for e:

$$e = \frac{1}{m}\left[-g_{\mu\nu}(x(\lambda))\dot{x}^{\mu}\dot{x}^{\nu}\right]^{1/2}, \tag{3.43}$$

where we can substitute e back, leading to $\hat{I} = I$. In that case, the equivalence between $\hat{I}$ and I has been demonstrated, which means that the einbein can be used to remove square root terms.

3.5.2 Brane Orbit

Using einbeins, as introduced in the previous subsection, we can expand (3.40) as

$$\begin{aligned} I_{\text{probe}} &= \int dt\{[\frac{1}{e}(-e^{6A}\dot{t}^2 + e^{2(B+2A)}\dot{y}^m\dot{y}^m) - \alpha e] - m\dot{t}e^C\}, \\ e &= \frac{2}{m}(e^{6A}\dot{t}^2 - e^{2(B+2A)}\dot{y}^m\dot{y}^m)^{\frac{1}{2}}, \end{aligned} \tag{3.44}$$

where we express the determinant with $\partial_i y^m = 0$. Then, we can derive the energy and angular momentum from (3.44) as follows:

$$\begin{aligned} \varepsilon &= -\frac{\partial\mathcal{L}}{\partial\dot{t}} = \left[\frac{1}{e}\left(2e^{6A}\dot{t}\right) + me^{3A}\right], \\ J &= \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = 2e^{3A}r^2\dot{\phi}\frac{1}{e}. \end{aligned} \tag{3.45}$$

Here the isotropic property $\dot{y}^m\dot{y}^m = \dot{r}^2 + r^2\dot{\phi}^2$ is used. Relabelling e as $\frac{2L}{m}$, we can combine the two equations in (3.45)[12]:

$$\dot{r}^2 = \frac{L^2}{m^2}e^{-3A}\left(\varepsilon^2 e^{-6A} - 2\varepsilon e^{-3A} - \frac{J^2}{r^2}e^{-3A}\right). \tag{3.46}$$

This is the equation for the isotropic radius of the probe brane, which is also known as the orbit of brane. Based on this equation, we can understand how the probe moves in the heavy brane background.

In this chapter, we investigated supergravity at $D = 11$ by determining its EoMs and conserved supercharges. We then applied a consistent truncation that links our theory with a scalar field. Based on the truncated action, we derived the Brane solution for electric and magnetic ansatzes. Finally, a multi-charge brane system was discussed that helps us to understand the motion of the probe brane.

[12]The detail derivations are in Section A.6

Chapter 4

Dimensional Reduction

In this chapter, we will explore the method of dimensional reduction. The genesis of this concept began with the proposition to extend our conventional four-dimensional (4D) spacetime into a five-dimensional (5D) framework, a theory initially put forth by Theodor Kaluza [17]. Oskar Klein [18] further developed this theory by suggesting that a fifth dimension could be conceptualised as a compactified circle, introducing what is now regarded as the quantum explanation of the extra dimension. This collection of ideas culminated in the well-known Kaluza–Klein theory. Concurrently, the technique of dimensional reduction emerged. Employing this method, we commence with a D-dimensional case. By compactifying on a circle, it becomes possible to perform a reduction from $D+1$ to D dimensions. This process can be iteratively applied to achieve a general reduction from $D+n$ to D dimensions.

After reviewing the fundamental methodology of dimensional reduction, we will delve into its corresponding symmetries, which will provide a better understanding of the scalar manifold in our theory.

4.1 Dimensional Reduction on S^1

It is convenient to start with a general dimension $D+1$. Considering the high-dimensional action (3.2) discussed in Chapter 3, we can split our theory into two parts: gravity and fields. Since all components depend on dimensions, we shall discuss them separately.

First, we need to consider the Ricci scalar, which is part of the gravity term. This can be calculated from the given metric term $g^{\mu\nu}$. In Klein's compactification idea, we reduce our dimension from $D+1$ to D with one dimension on a circle S^1. Before performing the detailed reduction, let us define some notation. Here, we use the capital letter $M(N)$ to represent the total $D+1$ index, the compactified dimension coordinate is written as z, and the rest of the coordinates as x. We express our $(D+1)$-dimensional metric as G_{MN}. Continuing with the Kaluza–Klein reduction, we assume that the compact circle is so small that one can remove the z dependence on the original metric. Now, our aim is to find the reduced D-dimensional metric from G_{MN}. One way to achieve this involves decomposing the metric index into two ranges, allowing us to expand G_{MN} as

$$G_{MN} = \begin{pmatrix} G_{\mu\nu} & G_{\mu z} \\ G_{z\nu} & G_{zz} \end{pmatrix}, \tag{4.1}$$

where the components can be relabelled to satisfy D-dimensional spacetime. Naively, one can match different components directly to lower-dimensional metrics; that is, we can let

$G_{\mu\nu}$ be $g_{\mu\nu}$. For $G_{\mu z}$ and G_{zz}, we can also label them as $\mathcal{A}_\mu$ and ϕ, respectively. However, these transformations cannot reveal the corresponding symmetry. A mixture of these compositions would be a better option [17]. Here, we rewrite the $(D+1)$-dimensional metric as

$$dS^2 = e^{2\alpha\phi} ds^2 + e^{2\beta\phi}(dz + \mathcal{A}_\mu dx^\mu)^2, \tag{4.2}$$

where we define the relationship between the $(D+1)$- and D-dimensional metrics as follows:

$$\begin{aligned} G_{\mu\nu} &= e^{2\alpha\phi} g_{\mu\nu} + e^{2\beta\phi}\mathcal{A}_\mu \mathcal{A}_\nu, \\ G_{\mu z} &= e^{2\beta\phi}\mathcal{A}_\mu, \\ G_{zz} &= e^{2\beta\phi}. \end{aligned} \tag{4.3}$$

We have already mentioned two methods to find the corresponding curvature. Here we use the spin connection method. Depending on the choice of metric, we can find our vielbein as follows:

$$E^a = e^{\alpha\phi} e^a, \quad E^z = e^{\beta\phi}(dz + \mathcal{A}_\mu dx^\mu). \tag{4.4}$$

For clarity, we use E and e to represent the $(D+1)$- and D-dimensional vielbeins, respectively. Then, we can derive the $(D+1)$ spin connection via (3.14) as follows:

$$\begin{aligned} & d(E^z) + \Omega^z{}_a \wedge E^a = 0, \\ \Rightarrow \quad & \beta\partial_\nu\phi dx^\nu \wedge E^z + e^{\beta\phi} d\mathcal{A}_\mu \wedge dx^\mu + \Omega^z{}_a \wedge E^a = 0, \\ \Rightarrow \quad & e^{-\alpha\phi}\beta\partial_a\phi E^a \wedge E^z + e^{\beta\phi-2\alpha\phi}\frac{1}{2}\mathcal{F}_{ab}E^a \wedge E^b + \Omega^z{}_a \wedge E^a = 0, \\ \Rightarrow \quad & \Omega^z{}_a = e^{-\alpha\phi}\beta\partial_a\phi E^z + \frac{1}{2}e^{\beta\phi-2\alpha\phi}\mathcal{F}_{ab}E^b. \end{aligned} \tag{4.5}$$

Again, we distinguish our spin connections by writing Ω and ω for $(D+1)$ and D dimensions. We also define a field strength $\mathcal{F}_{ab}$, which is the exterior derivative of the gauge field $\mathcal{A}_\mu$. Then, we can obtain another spin connection:

$$\begin{aligned} & dE^a + \Omega^a{}_b \wedge E^b + \Omega^a{}_z \wedge E^z = 0, \\ \Rightarrow \quad & \alpha e^{-\alpha\phi}\partial_b\phi E^b \wedge E^a - \omega^a{}_b \wedge E^b + \Omega^a{}_b \wedge E^b - \Omega^a{}_z \wedge E^z = 0, \\ \Rightarrow \quad & \Omega^a{}_b = \omega^a{}_b + \alpha e^{-\alpha\phi}\partial_b\phi E^a - \frac{1}{2}e^{\beta\phi-2\alpha\phi}\mathcal{F}^a{}_b E^z. \end{aligned} \tag{4.6}$$

From the derived 1-form spin connections, we can find the corresponding 2-form expressions $\mathcal{R}_{ab}$ and $\mathcal{R}_{zz}$ in terms of D-dimensional components:

$$\begin{aligned} \mathcal{R}_{ab} &= e^{2\alpha\phi}\left(R_{ab} - \alpha^2(D-1)(D-2)\partial_a\phi\partial_b\phi - \alpha\eta_{ab}\Box\phi\right) - \frac{1}{2}e^{2(\beta-2\alpha)\phi}F_a{}^c F_{bc}, \\ \mathcal{R}_{zz} &= (D-2)\alpha e^{-2\alpha\phi}\Box\phi + \frac{1}{4}e^{2(\beta-2\alpha)\phi}F_{ab}F^{ab}, \end{aligned} \tag{4.7}$$

where we can choose different values for α and β. The unfixed constants are restricted by the requirement of canonical normalisation. With this reduction, we gain a scalar field ϕ, which leads to a kinetic term in our final action. Thus, we need to set

$$\alpha^2 = \frac{1}{2(D-1)(D-2)}, \tag{4.8}$$

where we can find a $-\frac{1}{2}\partial_a\phi\partial_b\phi$ term in (4.7). Together with the reduced metric determinant $\sqrt{-g}$, we can then investigate the reduced gravity component with respect to a lower dimension:

$$\sqrt{-G}\mathcal{R} = e^{(\beta+(D-2)\alpha)\phi}\sqrt{-g}\left(R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4}e^{2(\beta-3\alpha)\phi}\mathcal{F}_{ab}\mathcal{F}^{ab}\right). \tag{4.9}$$

To keep the $\sqrt{-g}R$ form unchanged, we apply another constraint on α and β:

$$\beta = -(D-2)\alpha. \tag{4.10}$$

To summarise, we can rewrite the above gravity term using the forms introduced in Chapter 2:

$$I = \int R * 1 - \frac{1}{2} * d\phi \wedge d\phi - \frac{1}{2}e^{-2(D-1)\alpha\phi} * \mathcal{F}_{[2]} \wedge \mathcal{F}_{[2]}. \tag{4.11}$$

Now, let us consider the reduction for the gauge field and the antisymmetric field strength. In $(D+1)$ dimensions, we have a field strength $\hat{F}_{[n]}$ that can be expressed as $\hat{F}_{[n]} = d\hat{A}_{[n-1]}$, where $\hat{A}_{[n-1]}$ is the gauge potential. After compactification, we expect that there are two different potentials $A_{[n-1]}$ and $A_{[n-2]}$ in D dimensions that depend only on X. Thus, we can express the field strength in terms of the reduced gauge potential as [19]

$$\hat{F}_{[n]} = dA_{[n-1]} + A_{[n-2]} \wedge dz. \tag{4.12}$$

To maintain consistency with our metric reduction, we will introduce a new way to define the reduced field strength rather than the exterior derivative of the reduced gauged potential. This new choice links $\mathcal{A}_{[1]}$, defined in the metric, to our field strength (4.12) [19] as follows:

$$\begin{aligned}\hat{F}_{[n]} &= dA_{[n-1]} - dA_{[n-2]} \wedge \mathcal{A}_{[1]} + dA_{(n-2)} \wedge \left(dz + \mathcal{A}_{[1]}\right), \\ &= F_{[n]} + F_{[n-1]} \wedge \left(dz + \mathcal{A}_{[1]}\right),\end{aligned} \tag{4.13}$$

where the reduced field strength is defined as

$$F_{[n]} = dA_{[n-1]} - dA_{[n-2]} \wedge \mathcal{A}_{[1]}, \quad F_{[n-1]} = dA_{[n-2]}. \tag{4.14}$$

The consistent choice of reduced field strength allows us to expand the $(D+1)$-dimensional field strength with respect to the vielbein basis:

$$\begin{aligned}\hat{F}_{[n]} &= \frac{1}{n!}\hat{F}_{A_1\cdots A_n}E^{A_1} \wedge \cdots \wedge E^{A_n}, \\ &= \frac{e^{n\alpha\phi}}{n!}\hat{F}_{a_1\cdots a_n}e^{a_1} \wedge \cdots \wedge e^{a_n} + \frac{e^{((n-1)\alpha+\beta)\phi}}{(n-1)!}\hat{F}_{a_1\cdots a_{n-1}z}e^{a_1} \wedge \cdots \wedge e^{a_{n-1}} \wedge \left(dz + \mathcal{A}_{[1]}\right), \\ &= \frac{1}{n!}F_{a_1\cdots a_n}e^{a_1} \wedge \cdots \wedge e^{a_n} + \frac{1}{(n-1)!}F_{a_1\cdots a_{n-1}}e^{a_1} \wedge \cdots \wedge e^{a_{n-1}} \wedge \left(dz + \mathcal{A}_{[1]}\right).\end{aligned} \tag{4.15}$$

The capital letter A represents the total range of indices including a and z. After relabelling, we can find the relationship between the higher- and lower-dimensional field strengths:

$$\hat{F}_{a_1\cdots a_n} = e^{-n\alpha\phi}F_{a_1\cdots a_n}, \quad \hat{F}_{a_1\cdots a_{n-1}z} = e^{(D-n-1)\alpha\phi}F_{a_1\cdots a_{n-1}}. \tag{4.16}$$

We then consider the complete form for an antisymmetric field strength in our discussion. Again, we use forms to express the field part:

$$I = -\int \frac{1}{2} e^{-2(n-1)\alpha\phi} * F_{[n]} \wedge F_{[n]} + \frac{1}{2} e^{2(D-n)\alpha\phi} * F_{[n-1]} \wedge F_{[n-1]}. \tag{4.17}$$

An appropriate way to examine our reduction is to apply it to a real situation. Turning back to supergravity theory, we can start our S^1 compactification on the bosonic sector of 11-dimensional supergravity (3.2). Thus, we can reduce the 11-dimensional theory to 10 dimensions as follows:

$$\begin{aligned} & I_{11} = \int \left(R * 1 - \frac{1}{2} * F_{[4]} \wedge F_{[4]} \right) + \frac{1}{6} \int dA_{[3]} \wedge dA_{[3]} \wedge A_{[3]}, \\ \Rightarrow \quad & I_{10} = \int \left(R * 1 - \frac{1}{2} * d\phi \wedge d\phi - \frac{1}{2} e^{\frac{3}{2}\phi} * \mathcal{F}_{[2]} \wedge \mathcal{F}_{[2]} \right) \\ & \qquad - \left(\frac{1}{2} e^{\frac{1}{2}\phi} * F_{[4]} \wedge F_{[4]} + \frac{1}{2} e^{-\phi} * F_{[3]} \wedge F_{[3]} \right) + \frac{1}{2} \int dA_{[3]} \wedge dA_{[3]} \wedge A_{[2]}, \end{aligned} \tag{4.18}$$

where we write the previous 11-dimensional action in terms of forms. We also reduce the Chern–Simons term, which follows the field reduction relation only. As expected, the result after reduction shows the low-energy limit of the type IIA string theory. Although a lengthy calculation is required to determine all the reduction relations, the final results are straightforward and consistent with supergravity theory. At the end of this section, we derive the 10-dimensional theory. However, much work is required to obtain even lower-dimensional results. Naively, we can perform S^1 compactification repeatedly until we reach the target dimension. A generalised method to reduce multiple dimensions will be discussed in the next subsection.

4.2 Dimensional Reduction on T^n

In the previous discussion, we performed a compactification on an S^1 manifold, which reduced its dimensions by one. To repeat this process and obtain a general solution for n-dimensional reduction, we need to consider what kind of manifold to compactify. By continuously compactifying an S^1 manifold, we can obtain a combined manifold as $S^1 \times \cdots \times S^1$. Topologically, we define this manifold as a torus T^n. Thus, a reduction from $(D+n)$ to D dimensions is achieved by compactifying n extra dimensions on a T^n manifold.

Now, we can build on the previous discussion. In each reduction, we need to introduce a potential and a scalar term from the gravity component. For example, $\mathcal{A}^i_{[1]}$ and ϕ^i appear when we reduce our metric at step i. For the same reason, a p-form field and a $(p-1)$-form field are generated by reducing a p-form antisymmetric field. To illustrate these successive reductions, we shall use our $D = 11$ example again. Because we started from a maximal dimension, we can reduce from 11 to any positive dimension D on an $(11-D)$-torus. Due to the growing number of potentials, scalars and field strengths, we define a new label $i(jk)$ to represent the reduction process, which helps distinguish each component such that we can easily trace their descendants. For example, we can rewrite the above $D = 10$ action as follows:

$$I_{10} = \int \left(R * 1 - \frac{1}{2} * d\phi_1 \wedge d\phi_1 - \frac{1}{2} e^{\frac{3}{2}\phi_1} * \mathcal{F}^1_{[2]} \wedge \mathcal{F}^1_{[2]} \right) - \left(\frac{1}{2} e^{-\frac{1}{2}\phi_1} * F_{[4]} \wedge F_{[4]} + \frac{1}{2} e^{\phi_1} * F^1_{[3]} \wedge F^1_{[3]} \right) + \int \mathcal{L}_{CS}, \tag{4.19}$$

where we have $i = 1$ as we apply a 1-step reduction from 11 dimensions. To avoid redundancy, we will express the Lagrangian $\mathcal{L}$ in later discussions. The reduction of the Chern–Simons term $\mathcal{L}_{CS}$ will also be discussed at the end of this section.

Let us now return to the main Lagrangian reduction. A good way to generalise a theory is to start with some examples and compare them step by step. In that case, we can further reduce to $D = 9$. To see how our fields proliferate, we can focus on the change in coefficients in front of $*F_{[n]} \wedge F_{[n]}$. We have collected these in Table 4.1 for comparison.

	$D = 11$	$D = 10$	$D = 9$
$*F_{[4]} \wedge F_{[4]}$	1	$e^{-\frac{1}{2}\phi_1}$	$e^{-\frac{1}{2}\phi_1} e^{-\frac{3}{2\sqrt{7}}\phi_2}$
$*F_{[3]} \wedge F_{[3]}$	0	e^{ϕ_1}	$e^{-\frac{1}{2}\phi_1} e^{-\frac{5}{2\sqrt{7}}\phi_2} + e^{\phi_1} e^{-\frac{1}{\sqrt{7}}\phi_2}$
$*F_{[2]} \wedge F_{[2]}$	0	0	$e^{\phi_1} e^{\frac{3}{\sqrt{7}}\phi_2}$

Table 4.1: The coefficients of the field terms for $D = 9$, 10 and 11.

In Table 4.1, we ignore the factor $\frac{1}{2}$ because it is the same for all fields. We can also find that the number of 4-form terms does not increase. Instead, its coefficient is multiplied by an extra scalar term for each reduction. Other forms will have an additional term from higher-order forms. Usually, we call these scalar terms dilatons, and the corresponding values in front of ϕ are known as dilaton vectors. Let us first consider the 4-form dilaton vector. We can rewrite the coefficient term as a vector product $\vec{a} \cdot \vec{\phi}$, where $\vec{a}$ is dilaton vector:

$$\vec{a}_D = \left(\vec{a}_{D+1}, -2(n-1)\sqrt{\frac{1}{2(D-1)(D-2)}} \right), \tag{4.20}$$

which is derived from the previous field reduction relation (4.17). We also combine the scalars as $\vec{\phi} = (\phi_1, \phi_2, \cdots)$. The remaining field coefficients change according to a similar logic. Furthermore, the Kaluza–Klein potential also contributes to the field strength. To summarise all possible descendants of the field strength combination, we generalise a set of representations for dilaton vectors that depend on the value i, as shown in Table 4.2.

	$F_{[4]}$	$\mathcal{F}_{[2]}$
4-form	$\vec{a} = -\vec{g}$	N/A
3-form	$\vec{a}_i = \vec{f}_i - \vec{g}$	N/A
2-form	$\vec{a}_{ij} = \vec{f}_i + \vec{f}_j - \vec{g}$	$\vec{b}_i = -\vec{f}_i$
1-form	$\vec{a}_{ijk} = \vec{f}_i + \vec{f}_j + \vec{f}_k - \vec{g}$	$\vec{b}_{ij} = \vec{f}_i + \vec{f}_j$

Table 4.2: Dilaton vectors for different forms [19].

We define $\vec{g}$ and $\vec{f}_i$ to ensure that we are able to unify the complex expressions for different forms. Recall that we derived the dilaton vector for 4-forms in (4.20), which can be related to our new expression:

$$
\begin{aligned}
s_i &= 2 \times \sqrt{\frac{1}{2(10-i)(9-i)}},\\
\vec{g} &= 3 \times (s_1, s_2, \cdots, s_{11-D}),\\
\vec{f}_i &= (\underbrace{0, 0 \ldots, 0}_{i-1}, (10-1)s_i, s_{i+1}, \ldots, s_{11-D}),
\end{aligned}
\tag{4.21}
$$

where D is the target dimension. Using the fully defined vectors, let us now return to Table 4.2. The lower forms are found to be generated only from the higher forms at the start. In that case, the index $i(jk)$ in the lower forms expresses the need to satisfy $i < j < k$. For the empty value of $\mathcal{F}$, this is because one can only get 2-forms from the reduction of gravity terms. Then, the reduced Lagrangian can be expressed as [19]

$$
\begin{aligned}
\mathcal{L} = R * 1 &- \frac{1}{2} * d\vec{\phi} \wedge d\vec{\phi} - \frac{1}{2} e^{\vec{a}\cdot\vec{\phi}} * F_{[4]} \wedge F_{[4]} - \frac{1}{2}\sum_i e^{\vec{a}_i\cdot\vec{\phi}} * F^i_{[3]} \wedge F^i_{[3]}\\
&- \frac{1}{2}\sum_{i<j} e^{\vec{a}_{ij}\cdot\vec{\phi}} * F^{ij}_{[2]} \wedge F^{ij}_{[2]} - \frac{1}{2}\sum_i e^{\vec{b}_i\cdot\vec{\phi}} * \mathcal{F}^i_{[2]} \wedge \mathcal{F}^i_{[2]} - \frac{1}{2}\sum_{i<j<k} e^{\vec{a}_{ijk}\cdot\vec{\phi}} * F^{ijk}_{[1]} \wedge F^{ijk}_{[1]}\\
&- \frac{1}{2}\sum_{i<j} e^{\vec{b}_{ij}\cdot\vec{\phi}} * \mathcal{F}^{ij}_{[1]} \wedge \mathcal{F}^{ij}_{[1]} + \mathcal{L}_{CS}.
\end{aligned}
\tag{4.22}
$$

All the field strength combination reductions are summarised in the above equations. We also have a clear picture of a reduced Lagrangian. It is therefore time to discuss the metric reduction. Again, we will perform reductions on an 11-dimensional metric first:

$$
\begin{aligned}
ds^2_{11} =& e^{\frac{1}{6}\phi_1} ds^2_{10} + e^{-\frac{4}{3}\phi_1}\left(dz^1 + \mathcal{A}^1\right)^2,\\
\Rightarrow ds^2_{11} =& e^{\frac{1}{6}\phi_1} e^{\frac{1}{2\sqrt{7}}\phi_2} ds^2_9 + e^{\frac{1}{6}\phi_1} e^{\frac{\sqrt{7}}{2}\phi_2}\left(dz^2 + \mathcal{A}^2\right)^2\\
&+ e^{-\frac{4}{3}\phi_1}\left(dz^1 + \mathcal{A}^1 + \mathcal{A}^{12} \wedge dz^2\right)^2,
\end{aligned}
\tag{4.23}
$$

where dz^i is the reduced dimension. We also choose $\mathcal{A}^i$ as the first-order gauge potential and the reduced zeroth-order gauge potential as $\mathcal{A}^{ij}$. We can use the dilaton vector to express our metric. Considering a successive reduction from 11 to D dimensions, we can write our 11-dimensional metric as

$$
ds^2_{11} = e^{\frac{1}{3}\vec{g}\cdot\vec{\phi}} ds^2_D + \sum_i e^{(\frac{1}{3}\vec{g} - \vec{f}_i)\cdot\vec{\phi}}\left(dz^i + \mathcal{A}^i + \mathcal{A}^{ij} \wedge dz^j\right)^2,
\tag{4.24}
$$

where we can relabel some terms to simplify the expression:

$$
\begin{aligned}
\vec{\gamma}_i &= \frac{1}{6}\vec{g} - \frac{1}{2}\vec{f}_i,\\
h^i &= \left(dz^i + \mathcal{A}^i + \mathcal{A}^{ij} \wedge dz^j\right).
\end{aligned}
\tag{4.25}
$$

Next, we discuss the reduction of the field strength itself rather than its combination in the Lagrangian. We consider the gauge potential reduction as before and apply an exterior derivative. Let us start with a 3-form potential $A_{[3]}$ and reduce it to D dimensions:

$$
\hat{A}_{[3]} = A_{[3]} + A^i_{[2]} \wedge dz^i - \frac{1}{2!} A^{ij}_{[1]} \wedge dz^i \wedge dz^j - \frac{1}{3!} A^{ijk}_{[1]} \wedge dz^i \wedge dz^j \wedge dz^k,
\tag{4.26}
$$

where the minus sign comes from reordering the $i(jk)$ labels. Then, we can apply the exterior derivative. The form of the reduced field strength is defined as above. To link the current form to our metric, we will not represent F_n as dA_{n-1}. Instead, we will use a large number of terms to express our results as

$$\hat{F}_{[4]} = F_{[4]} + F^i_{[3]} \wedge h^i - \frac{1}{2}F^{ij}_{[2]} \wedge h^i \wedge h^j - \frac{1}{6}F^{ijk}_{[1]} \wedge h^i \wedge h^j \wedge h^k, \tag{4.27}$$

with a set of reduced field strengths, which is calculated in [19]:

$$\begin{aligned}
&F_4 = \tilde{F}_4 - \gamma^{ij}\tilde{F}^i_3 \wedge \mathcal{A}^j_1 - \frac{1}{2}\gamma^{ik}\gamma^{j\ell}\tilde{F}^{ij}_2 \wedge \mathcal{A}^k_1 \wedge \mathcal{A}^\ell_1 + \frac{1}{6}\gamma^{i\ell}\gamma^{jm}\gamma^{kn}\tilde{F}^{ijk}_1 \wedge \mathcal{A}^\ell_1 \wedge \mathcal{A}^m_1 \wedge \mathcal{A}^n_1,\\
&F^i_3 = \gamma^{ji}\tilde{F}^j_3 - \gamma^{ji}\gamma^{k\ell}\tilde{F}^{jk}_2 \wedge \mathcal{A}^\ell_1 - \frac{1}{2}\gamma^{ji}\gamma^{km}\gamma^{\ell n}\tilde{F}^{jk\ell}_1 \wedge \mathcal{A}^m_1 \wedge \mathcal{A}^n_1,\\
&F^{ij}_2 = \gamma^{ki}\gamma^{\ell j}\tilde{F}^{k\ell}_2 - \gamma^{ki}\gamma^{\ell j}\gamma^{mn}\tilde{F}^{k\ell m}_1 \wedge \mathcal{A}^n_1,\\
&F^{ijk}_1 = \gamma^{\ell i}\gamma^{mj}\gamma^{nk}\tilde{F}^{\ell mn}_1,\\
&\mathcal{F}^i_2 = \tilde{\mathcal{F}}^i_2 - \gamma^{jk}\tilde{\mathcal{F}}^{ij}_1 \wedge \mathcal{A}^k_1,\\
&\mathcal{F}^{ij}_1 = \gamma^{kj}\tilde{\mathcal{F}}^{ik}_1.
\end{aligned} \tag{4.28}$$

Using these expressions, we define $\tilde{F}_n = dA_{n-1}$. The new variable γ^{ij} comes from (4.25), and we can express dz^i in terms of h^i and the potential:

$$dz^i = [(1+\mathcal{A}_0)^{-1}]^{ij}(h^j - \mathcal{A}^j_1). \tag{4.29}$$

We can also define

$$\gamma^{ij} = [(1+\mathcal{A}_0)^{-1}]^{ij} = \delta^{ij} - \mathcal{A}^{ij} + \mathcal{A}^{il}\mathcal{A}^{lj} + \cdots . \tag{4.30}$$

Finally, we discuss the reduction of the Chern–Simons term, for which a general expression cannot be found. Instead, we must perform the derivation step by step. Since we have all the expressions for the reduced field, we can substitute them back with repeated calculations. Here, we show only a derivation for reduction from $D = 11$ to $D = 9$.[1] Reduction from 11 to 10 dimensions was discussed in the previous section, allowing us to start with the following expression:

$$\begin{aligned}
\mathcal{L}^{11}_{CS} &= \frac{1}{2}\left(dA_{[3]} \wedge dA_{[3]} \wedge A^i_{[2]}\right) \wedge dz^i\\
&= \frac{1}{2}\left(\tilde{F}_{[4]} \wedge \tilde{F}_{[4]} \wedge A^{ij}_{[1]} + 2\tilde{F}_{[4]} \wedge \tilde{F}^i_{[3]} \wedge \tilde{F}^j_{[3]} \wedge A_{[3]}\right) \wedge dz^j \wedge dz^i.
\end{aligned} \tag{4.31}$$

This can be further simplified by introducing

$$\epsilon^{ij} = 2dz^i \wedge dz^j, \tag{4.32}$$

which ensures the clarity of the expression at lower-dimensional terms. Until now, we have discussed the generalised torus compactification. If we want to perform a dimensional reduction on supergravity theory, we simply choose the target dimension and follow the results in this section. This saves a great deal of time when we research specific dimensions. In the discussion about reduced components of the Lagrangian, we did not mention another important part of our theory, which is its symmetry. In the next subsection, we will talk about the symmetry involved during dimensional reduction.

[1]The full reduced terms from 10 to 2 dimensions are presented in the Section B.1.

4.3 Symmetries in Dimensional Reduction

To understand a theory, it is not enough to calculate the expressions without examining the underlying symmetries. Especially in dimensional reduction, the scalars or dilatons will increase at each reduction. At the same time, we will encounter 0-form gauge potentials, called “axions”, which can also be scalar terms. To solve these scalar terms in the Lagrangian, it is important to identify their symmetries. In this subsection, we will first investigate the simple S^1 compactification symmetries. Then, torus reduction will be discussed, which links to the concept of scalar coset Lagrangians.

4.3.1 S^1 Reduction Symmetry

Let us turn back to the start of dimensional reduction. We investigate the simplest S^1 reduction symmetry first. Considering higher- and lower-dimensional theories separately, they must be covariant for their own coordinates. Then, we can link these theories together via our reduced metric ansatz, which leads to the fact that the symmetries of lower dimensions are composed of higher ones [7]. Specifically, the reduced D-dimensional theory will have a local gauge symmetry and a shift symmetry in addition to the general coordinate symmetry.

We shall start carefully with a general coordinate transformation from a $(D+1)$-dimensional theory [20]:

$$\delta X^M = -\Xi^M, \quad \delta G_{MN} = \Xi^P \partial_P G_{MN} + G_{PN}\partial_M \Xi^M + G_{MP}\partial_N \Xi^P. \tag{4.33}$$

Since the $(D+1)$-dimensional theory has general coordinate symmetry, its metric must satisfy the above infinitesimal transform via Ξ^M, which are functions of the total coordinates X^M. We can then apply the reduction method to these transformations. To follow the metric ansatz introduced above, the simplest forms we can take are

$$\Xi^\mu = \xi^\mu(x), \quad \Xi^z = cz + \lambda(x), \tag{4.34}$$

where we express the transformations into two ranges with D dimension indices μ and an extra dimension index z. The corresponding functions $\xi^\mu(x)$ are the ansatz for D dimensions. By choosing a different index for $M(N)$ and substituting[2] the results back into (4.1), we obtain the following results:

$$\begin{aligned}
&\delta\phi = \xi^\rho \partial_\rho \phi, \\
&\delta\mathcal{A}_{mu} = \xi^\rho \partial_\rho \mathcal{A}_{mu} + \mathcal{A}_\rho \partial_\mu \xi^\rho + \partial_\mu \lambda, \\
&\delta g_{\mu\nu} = \xi^\rho \partial_\rho g_{\mu\nu} + g_{\rho\nu}\partial_\mu \xi^\rho + g_{\mu\rho}\partial_\nu \xi^\rho,
\end{aligned} \tag{4.35}$$

where we set $c = 0$. The first equation illustrates that ϕ follows the local coordinate symmetry as a scalar. It is also invariant under a $U(1)$ transformation with respect to λ. For the second equation, we find that $\mathcal{A}_{mu}$ satisfies both the general coordinate transformation and the $U(1)$ transform. The last equation shows that the metric term transforms appropriately in both symmetries. Thus, we have shown that the reduced D-dimensional theory preserves the local symmetries.

On the other hand, it is necessary to discuss the non-vanishing constant situation in which we set c in Ξ^z non-zero. This leads to another important symmetry in our

[2] The detailed calculations are presented in Section B.2.

dimensional reduction, which is termed dilaton shift symmetry. It can be achieved by a shift transformation on ϕ with a suitable scaling transformation [7] on $\mathcal{A}_\mu$:

$$\phi \to \phi + c, \quad \mathcal{A}_\mu \to e^{c(D-1)\alpha}\mathcal{A}_\mu. \tag{4.36}$$

To justify this symmetry, we require more information about our higher-dimensional theory. In the $(D+1)$-dimensional Lagrangian, we can easily find an EoM[3] that has an additional global symmetry:

$$G_{MN} \to k^2 G_{MN}, \tag{4.37}$$

which keeps the EoM unchanged. Therefore, one can apply this global symmetry, which is independent of the general coordinate x, and obtain the following corresponding infinitesimal form:

$$\delta G_{MN} = c\delta^z_M G_{zN} + c\delta^z_N G_{Mz} + 2aG_{MN}. \tag{4.38}$$

Applying the same process as was used for the local symmetry, we can substitute the metric back into our ansatz, which gives the following results:[4]

$$\beta\delta\phi = a + c, \quad \delta\mathcal{A}_\mu = -c\mathcal{A}_\mu, \quad \delta g_{\mu\nu} = 2ag_{\mu\nu} - 2\alpha g_{\mu\nu}\delta\phi. \tag{4.39}$$

Notably, our lower-dimensional metric is also invariant under such a scaling transform. Thus, we need to maintain $\delta g_{\mu\nu}$ at zero via a suitable relation between the constants. While relabelling these constants,[5] one can easily show that the shift symmetry in (4.36) is satisfied.

4.3.2 T^2 Reduction Symmetry

In the previous subsection, we discussed S^1 reduction, which contains only one dilaton. However, a general case always has two types of scalar, namely dilatons and axions. To examine this multi-scalar case, we need to consider more than one step of reduction, which gives an axion. Thus, a T^2 compactification is a good example to investigate.

Beginning with the gravity term of our theory, we can find the Lagrangian expression and reduced metric term via (4.22) and (4.24). Unlike the 1-step reduction, we now have an $\mathcal{A}^{12}$ term, which has a 0-form potential and can be represented as χ. With some relabelling and substitutions, we obtain the most appropriate expression of the Lagrangian for our discussion as follows:

$$\mathcal{L} = R{*}1 - \frac{1}{2}{*}d\varphi\wedge d\varphi - \frac{1}{2}{*}d\phi\wedge d\phi - \frac{1}{2}e^{\phi+q\varphi}{*}\mathcal{F}^1_{(2)}\wedge\mathcal{F}^1_{(2)} - \frac{1}{2}e^{-\phi+q\varphi}{*}\mathcal{F}^2_{(2)}\wedge\mathcal{F}^2_{(2)} - \frac{1}{2}e^{2\phi}{*}d\chi\wedge d\chi, \tag{4.40}$$

where we have $q = \sqrt{D/(D-2)}$. Its corresponding metric is as follows:

$$ds^2_{D+2} = e^{-\frac{2}{\sqrt{D(D-2)}}\varphi}ds^2_D + e^{\sqrt{(D-2)/D}\varphi}\left(e^{\phi}\left(dz_1 + \mathcal{A}^1 + \chi dz_2\right)^2 + e^{-\phi}\left(dz_2 + \mathcal{A}^2\right)^2\right). \tag{4.41}$$

From the structure of this metric, we find that the original $(D+2)$ metric is split into a D-dimensional metric and a torus metric. These three different scalar terms have various properties based on their positions in the metric. φ appears as an overall factor

[3] By varying the metric term, one can obtain $R_{MN} - \frac{1}{2}RG_{MN} = 0$.

[4] All the substitutions are shown in Section B.2.

[5] One can find the relation $a = -c/(D-1)$.

in front of the torus metric, which can influence the volume of the final torus. This factor can be easily isolated, which makes it a common scalar term as discussed above. Thus, we will have a shift symmetry on φ. Next, we need to consider the remaining scalars ϕ and χ, which determine the shape of the compactified torus. By varying ϕ, one can change the radii of the two circles. The other value χ determines their relative angles. The related Lagrangian for the dilaton and axion is

$$\mathcal{L}_{(\phi,\chi)} \equiv -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}e^{2\phi}(\partial\chi)^2 = -\frac{\partial\tau\cdot\partial\bar{\tau}}{2\tau_2^2}, \tag{4.42}$$

We can combine these terms via a complex field $\tau = \chi + ie^{-\phi}$, where τ_2 is the imaginary part of the field. This newly defined field helps us to evaluate the underlying symmetry and can be shown by applying a fractional linear transformation:

$$\tau \longrightarrow \frac{a\tau + b}{c\tau + d}, \tag{4.43}$$

We then restrict our constants with $ad - bc = 1$ such that the Lagrangian is unchanged. A similar operation is used in [21] to show T^7 symmetry. We can then rewrite this as the following matrix:

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{4.44}$$

which forms an $SL(2,\mathbb{R})$ group.

Thus, we have an $SL(2,\mathbb{R})$ symmetry for the dilaton–axion system, which has non-linear symmetry. Together with the shift symmetry, which is known as $\mathbb{R}$, we have a full $GL(2,\mathbb{R})$ symmetry for the gravity part of the theory. Now, we need to ensure that the same symmetry can be observed for the field part of the theory. Before transforming the field term, we can define a new $\mathcal{A}^1$ as $\mathcal{A}^1 + \chi\mathcal{A}^2$. Comparing with the original $\mathcal{F}^1$ expression via (4.28), we find that

$$\mathcal{F}^1 = d\mathcal{A}^1 - d\chi \wedge \mathcal{A}^2 \longrightarrow \mathcal{F}^1 = d\mathcal{A}^1 + \chi d\mathcal{A}^2, \tag{4.45}$$

Consequently, the field components now all depend on the derivatives of the potential. Thus, we can only check the transformation on gauge potentials, which is straightforward:

$$\begin{pmatrix} \mathcal{A}^2 \\ \mathcal{A}^1 \end{pmatrix} \longrightarrow \left(\Lambda^T\right)^{-1} \begin{pmatrix} \mathcal{A}^2 \\ \mathcal{A}^1 \end{pmatrix}, \tag{4.46}$$

and maintains the Lagrangian invariant as expected. Furthermore, we can show that the transformation here is linear, unlike the dilaton–axion symmetry. Extending our current discussion to a T^n case, we can naively deduce that there is a global $GL(n,\mathbb{R})$ symmetry. In real situations, this is not always true. Detailed discussions for each dimension are presented in [20]. To characterise the global symmetry, we need to further examine the scalar Lagrangian. Usually, the global symmetry of a reduced dimensional theory can be confirmed by checking only its corresponding scalar Lagrangian. Thus, we shall go through the scalar part again but with a generalised method in the next subsection. This will lead us to the concept of the coset space of a scalar Lagrangian.

4.3.3 Global Symmetries as Scalar Cosets

We can now start from the $SL(2, \mathbb{R})$ symmetry group. We can further investigate its Lie algebra [20] with its related generators:

$$[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = H. \tag{4.47}$$

We can express these as 2×2 matrices:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{4.48}$$

To link this symmetry to our Lagrangian, we can begin by defining [22]

$$\mathcal{V} = e^{\frac{1}{2}\phi H} e^{\chi E}, \quad \mathcal{M} = \mathcal{V}^T \mathcal{V}, \tag{4.49}$$

where ϕ and χ are the scalar fields from the previous subsection. Using these definitions, we can rewrite the scalar Lagrangian as follows:

$$\mathcal{L} = \frac{1}{4} \operatorname{tr} \left(\partial \mathcal{M}^{-1} \partial \mathcal{M} \right) = -\frac{1}{2} (\partial \phi)^2 - \frac{1}{2} e^{2\phi} (\partial \chi)^2, \tag{4.50}$$

which will directly reveal the invariance when we apply the transformation. Again, we can use the Λ matrix defined above, which allows our matrices to be transformed as follows:

$$\begin{aligned} \mathcal{V} &\longrightarrow \mathcal{V}\Lambda, \\ \mathcal{M} &\longrightarrow \Lambda^T \mathcal{M} \Lambda, \end{aligned} \tag{4.51}$$

The corresponding scalar Lagrangian is invariant since we take a trace of the matrix $\mathcal{M}$. However, one can show that the actual transformation is forbidden as it breaks the original form of $\mathcal{V}$. At the same time, we can regard the matrix $\mathcal{V}$ as the representation of the dilaton and axion. In the previous subsection, we transformed them based on a defined complex field τ. Here, we can define a local transformation $\mathcal{O}$ that keeps the form of $\mathcal{V}$ while it interacts with $SL(2, \mathbb{R})$ symmetries. After this modification, we can transform our matrix as follows:

$$\mathcal{V} \longrightarrow \mathcal{O}\mathcal{V}\Lambda, \tag{4.52}$$

which requires $\mathcal{O}$ to be orthogonal for the Lagrangian to maintain its invariance. This gives a local $O(2)$ symmetry at the end.

To summarise the above discussion, the global symmetry links all the values of ϕ and χ together on a scalar manifold. Simultaneously, an $\mathcal{O}(2)$ transformation is applied as a correction to keep the form unchanged. In this case, we form a coset space $SL(2, \mathbb{R})/\mathcal{O}(2)$, which can be regarded as the manifold of the dilaton–axion scalar system. As we further reduce the dimension, we will encounter more scalars, which lead to different global symmetries and correction transformations for each dimension. These are summarised in Table 4.3.

In this chapter, we have discussed the means by which we can obtain a type IIA string theory in the framework of supergravity, which can be regarded as a S^1 compactification. We then expanded this idea to obtain a general torus reduction. After finding the reduced forms of both the gravity and field terms in the Lagrangian, their symmetries were investigated. Finally, we summarised the scalar Lagrangian symmetry that makes all scalars on a coset space determined by the global symmetry and a suitable correction. However, there are still many important topics in dimensional reduction that we have not covered. One might compactify our theory in manifolds other than T^n, for example S^n [23], [24] or a Calabi–Yau manifold [25].

	G	K
$D = 8$	$SL(3, \mathbb{R}) \times SL(2, \mathbb{R})$	$SO(3) \times SO(2)$
$D = 7$	$SL(5, \mathbb{R})$	$SO(5)$
$D = 6$	$O(5, 5)$	$O(5) \times O(5)$
$D = 5$	$E_{6(+6)}$	$USp(8)$
$D = 4$	$E_{7(+7)}$	$SU(8)$
$D = 3$	$E_{8(+8)}$	$SO(16)$

Table 4.3: Coset space for D dimensions [22]

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Chapter 5

T-Duality

T-duality is a special duality between two string theories and was first introduced in [26]. Subsequently, in the mid-1990s, this duality was extended into superstring theory, which helped to link the type IIA and type IIB string theories together [27], [28]. It now acts as a cornerstone of M-theory. To understand T-duality, one can imagine a propagating string on a circle with radius R, which is equal to one with radius $1/R$. Moreover, we can relate two physical quantities in different theories with a suitable T-duality transformation. The most famous example is that momentum can be regarded as a dual representation of the winding number in another theory. In the simplest case, the winding number can be defined as the number of turns required for a string to be wrapped around a cylinder. In the following, we aim to find a T-duality between the type IIA and type IIB supergravity theories.

In the previous chapter, we successfully derived type IIA supergravity from the $D = 11$ theory. However, one cannot repeat this process for the type IIB theory. This is mainly because the chiral properties of the type IIB theory requires a self-duality condition. Additionally, chirality acts as a barrier between two theories, making it impossible to apply T-duality at $D = 10$. To find a suitable expression, we can write something that is similar to type IIA but obeys the self-duality condition. In [29], a different scaling factor is used for dimension reduction on S^1. Although our earlier method can help us to obtain the reduction quickly, it is necessary to compare the two theories, which requires a clear way to illustrate each term in the actions. Using this new scaling, we can find that both the type IIA and IIB theories have the same sector, namely the Neveu–Schwarz (NS) sector:

$$S_{NS} = \int e^{2\varphi} \left(R * 1 + 4 * d\varphi \wedge d\varphi - \frac{1}{2} * H_{[3]} \wedge H_{[3]} \right), \tag{5.1}$$

where φ is the scaled scalar and $H_{[3]}$ represents the field strength. The next component is the Ramond (R) sector, which contains several antisymmetric field strength terms that are different for each theory. Then, we need to introduce two sets of Chern–Simons (CS) terms to complete the actions. Here, we have R sector expressions of the action as follows:

$$\begin{aligned} S_R^{IIA} &= -\frac{1}{2} \int \left(*\tilde{F}_{[2]} \wedge \tilde{F}_{[2]} + *F_{[4]} \wedge F_{[4]} \right), \\ S_R^{IIB} &= -\frac{1}{2} \int \left(*\tilde{F}_{[1]} \wedge \tilde{F}_{[1]} + *F_{[3]} \wedge F_{[3]} + \frac{1}{2} * F_{[5]} \wedge F_{[5]} \right), \end{aligned} \tag{5.2}$$

where $F_{[5]}$ is a self-dual form in $D = 10$, $\tilde{F}_{[n]}$ is the field strength defined by $dA_{[n-1]}$, and $F_{[n]}$ is the modified field strength, which is similar to the relation (4.28).

Now, let us consider the simplest case for the type IIB theory. First, the identical NS sectors lead to the same expression for type IIA and type IIB after dimensional reduction. For the R sector, one can set the 5-form to zero since it satisfies the self-duality condition; a similar idea is discussed in [28]. In this case, the action in the R sector leaves 1-form and 3-form field strengths. Following the reduction rule (4.17), we obtain a 3-form term and a 2-form term from the reduction of $F_{[3]}$. Note that we do not obtain the reduced $\tilde{F}_{[1]}$ as it is formed from a 0-form potential. Recalling that we have already discussed the reduced type IIA theory in the T^2 compactification case, the results include 1- to 4-form field strengths. Thus, if it is necessary for all the terms in type IIA and IIB to match, the 4-forms after reduction must vanish [28].

Therefore, we have shown that the reduced type IIA and IIB theories in $D = 9$ are actually the same theory. Furthermore, two 2-forms in type IIA and type IIB are connected via a global symmetry, which demonstrates the existence of T-duality between the two theories. In other words, we can generate two different 10-dimensional supergravity theories from one theory in $D = 9$. In [27], this dimension-raising process is termed "decompactification". Geometrically, we can restate our example of T-duality as two theories that are compactified on an S^1 manifold from $D = 10$. Thus, the radii defined via dimensional reductions need to satisfy [30]

$$R_{IIA} = \alpha'/R_{IIB}, \tag{5.3}$$

where the labels denote the different theories.

In this chapter, we considered the T-duality between the type IIA and type IIB theories in nine dimensions, which links two different superstring theories together. With this implication of T-duality, we gained a deeper understanding of string theory and supergravity. Furthermore, the link between these types of superstring theory gives us M-theory and enables us to consider the underlying nature of duality.

Chapter 6

Conclusion

In this dissertation, we have extensively explored the theory of supergravity. Initially, we introduced the basic definitions for form representations and related operations, providing a clear framework for expressing our calculations and results in the subsequent sections. Following this foundational setup, we focused on a "brane", which is a commonly used dynamic object in supergravity. Starting with a consistent truncation from maximal supergravity at $D = 11$, we derived an action for branes in general D dimensions. Consequently, we discovered brane solutions based on the EoMs derived from the variation of the action. During this process, methods such as the vielbein approach and Weyl transformations were employed to construct the curvature. These methods are pivotal in theoretical physics.

We then set $D = 11$ to identify specific brane solutions in maximal supergravity, leading to the discovery of M2-branes and M5-branes. Concluding the discussion of individual branes, Chapter 3 considered a multi-brane scenario, providing a systematic approach to analysing the trajectories of moving branes.

Following the discussion of brane solutions, we revisited the truncation theory, commonly known as dimensional reduction. Through a detailed examination of this method, we successfully derived the Lagrangian term for type IIA supergravity in $D = 10$. Extending this concept, we achieved continuous reduction with T^n compactification. It is noteworthy that scalars tend to accumulate during dimensional reduction. Consequently, we focused on scalar symmetries after identifying all the reduced forms of the Lagrangian. Typically, such scalar symmetries are applicable to the entire system. By examining these symmetries in our previous discussions, we identified a coset space that describes the manifold for all scalars. Each reduction introduces a distinct coset space, enhancing our understanding of the scalar Lagrangian in supergravity.

Towards the end of this dissertation, we delved into another critical aspect of supergravity, T-duality, particularly that between type IIA and IIB supergravity. Initially, we introduced the type IIB theory, characterised by a specific self-duality condition, which precludes its derivation from maximal supergravity. We also discovered that T-duality cannot be directly applied at $D = 10$ due to the chirality of the type IIB theory. To circumvent this problem, we again employed dimensional reduction, equalising the two theories in $D = 9$. This reduction process gives rise to a global symmetry, known as T-duality. Serving as a pivotal bridge linking the two theories, T-duality plays an integral role in our understanding of M-theory.

Although this dissertation has presented a wide-ranging discussion from brane solutions to T-duality, it is important to acknowledge that our exploration covers only a portion of the vast landscape of string theory and supergravity. There remains a substantial portion of the theory that has not been discussed, presenting ample opportunity for further investigation. One possible extension is to continue our examination of branes. Since we have only discussed the maximal supergravity condition, there are still many other types of branes in different dimensions left to explore. In particular, a D-brane, which satisfies the Dirichlet boundary condition, can be used in string theory and its implications include a new duality termed Ads/CFT [31]. This correspondence offers a powerful framework for relating gravitational theories in anti-de Sitter spaces to conformal field theories with one less dimension, which could provide deeper insights into quantum gravity. Additionally, mirror symmetry [32], as another interesting duality, merits further exploration. This could be an extension for our compactification chapter as one can reduce the dimension of a Calabi–Yau manifold leading to mirror symmetry. In particular, two completely different Calabi–Yau manifolds can be linked to each other through this symmetry.

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Appendix A

Brane Solution and Related Calculations

A.1 Derivation of EoM for 11-Dimensional Supergravity

Let us vary (3.2):

$$
\begin{aligned}
\delta I_{11} &= \int d^{11}x\delta\left\{\sqrt{-g}\left(R-\frac{1}{48}F_{[4]}^2\right)\right\}-\frac{1}{6}\int\delta\left(F_{[4]}\wedge F_{[4]}\wedge F_{[3]}\right)\\
&= \int d^{11}x\delta\{\sqrt{-g}R\}-\int d^{11}x\delta\left\{\sqrt{-g}\frac{1}{48}F_{[4]}^2\right\}-\frac{1}{6}\int\delta\left(F_{[4]}\wedge F_{[4]}\wedge F_{[3]}\right).
\end{aligned}
\tag{A.1}
$$

The first term vanishes as it is a total derivative form. We will therefore focus on the last two terms. We start with the last term, which is mainly a variation over the wedge product, and for which (2.4) and (2.5) can be used as a means of explanation:

$$
\begin{aligned}
&\delta\left(F_{[4]}\wedge F_{[4]}\wedge A_{[3]}\right)\\
=&\delta F_{[4]}\wedge F_{[4]}\wedge A_{[3]}+F_{[4]}\wedge\delta F_{[4]}\wedge A_{[3]}+F_{[4]}\wedge F_{[4]}\wedge\delta A_{[3]}\\
=&3F_{[4]}\wedge F_{[4]}\wedge\delta A_{[3]}.
\end{aligned}
\tag{A.2}
$$

This is the last term for our final result in (3.4). Hence, the rest of the work is to derive (3.3) and then show that the coefficient is correct. Continuing with (A.1) and turning this into tensor form via (2.1),

$$
\begin{aligned}
&\int d^{11}x\sqrt{-g}\frac{1}{48}F_{mnpq}F^{mnpq}\\
=&\frac{1}{48}\int F_{mnpq}F^{mnpq}\epsilon.
\end{aligned}
\tag{A.3}
$$

At the same time, we can expend $F_{[4]}\wedge *F_{[4]}$ as an 11-form component:

$$
F_{[4]}\wedge *F_{[4]}=\frac{z}{11!}\epsilon_{j_1}\dots{}_{j_{11}}\,dx^{j_1}\wedge dx^{j_2}\wedge\cdots dx^{j_{11}},
\tag{A.4}
$$

where z is the function that satisfies

$$
\int F_{[4]}\wedge *F_{[4]}=\int\sqrt{-g}d^{11}xz.
\tag{A.5}
$$

We can also express (A.4) based on the wedge product as follows:

$$F_{[4]} \wedge *F_{[4]} = \frac{11!}{4!7!} F_{[i_1 \cdots i_4} (*F)_{i_5 \cdots i_{11}]}, \tag{A.6}$$

where the expression of the Hodge dual can be found as follows:

$$*F_{[4]} = \frac{1}{4!} \epsilon_{\mu_1 \cdots \mu_1 v_8 \cdots v_{11}} F^{v_8 \cdots v_{11}}. \tag{A.7}$$

The following substitution leads us to the final results:

$$\begin{aligned} z\epsilon_{i_1 \cdots i_{11}} &= \frac{11!}{4!7!} F_{[i_1 \cdots i_4} \left(\frac{1}{4!} \epsilon_{i_5 \ldots i_{11}] v_8 \cdots v_{11}} F^{v_8 \cdots v_{11}} \right) \\ \Rightarrow \quad -11! z &= \left(\frac{11!}{4!7!} F_{i_1 \cdots i_4} \frac{1}{4!} \epsilon_{i_5 \ldots i_{11} v_8 \ldots v_{11}} F^{v_8 \cdots v_{11}} \right) \epsilon^{i_1 \cdots i_{11}} \\ \Rightarrow \quad z &= \frac{1}{4!} F_{i_1 \cdots i_4} F^{i_1 \cdots i_4}. \end{aligned} \tag{A.8}$$

Here, we have used a contraction of the Levi–Civita tensor via (2.12):

$$\epsilon_{i_5 \cdots i_{11} v_8 \cdots v_{11}} \epsilon^{i_1 \cdots i_{11}} = (-1) 4! 7! \delta^{i_1 \cdots i_4}_{v_8 \cdots v_{11}}. \tag{A.9}$$

Now, we prove the equality of the middle terms in (A.1) and (A.4). Then, we can show that the corresponding variation is as follows:

$$\begin{aligned} &\delta \left(F_{[4]} \wedge *F_{[4]} \right) \\ =&\delta F_{[4]} \wedge *F_{[4]} + F_{[4]} \wedge \delta \left(*F_{[4]} \right) \\ =&2 \left(\delta F_{[4]} \wedge *F_{[4]} \right) \\ =&2 \left(\delta dA_{[3]} \wedge *F_{[4]} \right) \\ =&2 \left(\delta A_{[3]} \wedge d * F_{[4]} \right). \end{aligned} \tag{A.10}$$

This leads to the correct factor and is thus a proof of (3.4).

A.2 Derivation of EoMs for D-Dimensional Supergravity

A.2.1 Variation with Respect to the Metric

We here discuss the variation with respect to the metric g_{MN}. Considering the possible variables that contain the metric, we have $\sqrt{-g}$, g^{MN} and g. We first perform the variation on these values:

$$\begin{aligned} \delta g &= g^{MN} g \delta \left(g_{MN} \right), \\ \delta(\sqrt{-g}) &= \frac{1}{2} \sqrt{-g} g^{MN} \delta \left(g_{MN} \right), \\ \delta \left(g^{RS} \right) &= -g^{RM} g^{SN} \delta \left(g_{MN} \right). \end{aligned} \tag{A.11}$$

Starting from the first term in (3.9),

$$\begin{aligned}
&\delta \int d^D x \sqrt{-g} \left(g^{MN} R_{MN}\right) \\
&= \int d^D x \left(\delta(\sqrt{-g}) \left(g^{MN} R_{MN}\right) + \sqrt{-g} \left(\delta g^{MN}\right) R_{MN} + \sqrt{-g} g^{MN} \delta \left(R_{MN}\right)\right) \\
&= \int d^D x \left(\frac{1}{2} \sqrt{-g} g^{MN} R + \sqrt{-g} \left(-g^{RM} g^{SN}\right) R_{RS} + 0\right) \delta \left(g_{MN}\right).
\end{aligned} \tag{A.12}$$

The result leads to the Einstein equation:

$$-R^{MN} + \frac{1}{2} g^{MN} R = 0 \tag{A.13}$$

We then apply a variation to the second term:

$$\begin{aligned}
&\delta \int d^D x \sqrt{-g} \left(-\frac{1}{2} \nabla_M \phi \nabla^M \phi\right) \\
&= \int d^D x \left(\delta(\sqrt{-g}) \left(-\frac{1}{2} \nabla_M \phi \nabla^M \phi\right) + \sqrt{-g} \delta \left(-\frac{1}{2} \nabla_A \phi \nabla^A \phi\right)\right) \\
&= \int d^D x \left(\frac{1}{2} \sqrt{-g} g^{MN} \left(-\frac{1}{2} \nabla_M \phi \nabla^M \phi\right) + \sqrt{-g} \delta \left(-\frac{1}{2} \nabla_R \phi \nabla_S \phi g^{RS}\right)\right) \\
&= \int d^D x \left[\frac{1}{2} \sqrt{-g} g^{MN} \left(-\frac{1}{2} \nabla_M \phi \nabla^M \phi\right) + \sqrt{-g} \left(-\frac{1}{2} \nabla_R \phi \nabla_S \phi \left(-g^{RM} g^{SN}\right)\right)\right] \delta \left(g_{MN}\right)
\end{aligned} \tag{A.14}$$

Combining the results from (A.12) and (A.14), we obtain

$$-R_{MN} + \frac{1}{2} g_{MN} R + \frac{1}{2} \nabla_M \phi \nabla_N \phi - \frac{1}{4} g_{MN} \nabla_A \phi \nabla^A \phi = 0 \tag{A.15}$$

Multiplying (A.15) by g^{MN} and taking the trace leads to

$$\begin{aligned}
&R - \frac{D}{2} g_{MN} - \frac{1}{2} \nabla_R \phi \nabla^R \phi + \frac{D}{4} \nabla_A \phi \nabla^A \phi = 0 \\
\Rightarrow \quad &\left(1 - \frac{D}{2}\right) R - \frac{1}{2} \left(1 - \frac{R}{2}\right) \nabla_R \phi \nabla^R \phi = 0 \\
\Rightarrow \quad &R = \frac{1}{2} \nabla_R \phi \nabla^R \phi,
\end{aligned} \tag{A.16}$$

where we have $\mathrm{Tr}\left(g_{MN} g^{MN}\right) = D$. Hence, (A.15) can be simplified as follows:

$$\begin{aligned}
&R_{MN} = -\frac{1}{4} g_{MN} \nabla_A \phi \nabla^A \phi - \frac{1}{2} \nabla_M \phi \nabla_N \phi + \frac{1}{4} g_{MN} \nabla_A \phi \nabla^A \phi \\
&\Rightarrow R_{MN} = \frac{1}{2} \nabla_M \phi \nabla_N \phi.
\end{aligned} \tag{A.17}$$

Now, we focus on the last term, which we can expand as

$$F_{[n]}^2 = F_{R_1} \ldots F_{S_1} \ldots g^{R_1 S_1} g^{R_2 S_2} \ldots g^{R_n S_n}. \tag{A.18}$$

Applying variation to this form yields

$$\begin{aligned}
\frac{\delta\left(F^2\right)}{\delta\left(g_{MN}\right)} &= -g^{MR_1} g^{NS_1} F_{R_1 \ldots} F_{S_1 \ldots} g^{R_2 S_2} \ldots g^{R_n S_n} + \ldots \\
&= -F^M_{R_2 \ldots} F^N_{S_2 \ldots} g^{R_2 S_2} \ldots g^{R_n S_n} n.
\end{aligned} \tag{A.19}$$

Here, we apply the antisymmetric property of the strength tensor F, which gives the n in the result. Again considering the variation in the last term, we find that

$$
\begin{aligned}
&\int d^D x \sqrt{-g}\left(-\frac{e^{a\phi}}{2n!}F_{[n]}^2\right) \\
=&\int d^D x \left(\frac{1}{2}\sqrt{-g}g^{MN}\delta\left(g_{MN}\right)\left(-\frac{e^{a\phi}}{2n!}F_{[n]}^2\right) - \sqrt{-g}\frac{e^{a\phi}}{2n!}\delta\left(F_{[n]}^2\right)\right) \\
=&\int d^D x \left(\frac{1}{2}\sqrt{-g}g^{MN}\cdot\left(-\frac{e^{a\phi}}{2n!}F_{[n]}^2\right) - \sqrt{-g}\frac{e^{a\phi}}{2n!}n\left(-F^{M\cdots}F^{N}\cdots\right)\right)\delta\left(g_{MN}\right).
\end{aligned}
\tag{A.20}
$$

Combining all results, we obtain the following:

$$
\begin{aligned}
&R_{MN} - \frac{1}{2}g_{MN}R - \frac{1}{2}\nabla_M\phi\nabla_N\phi + \frac{1}{4}g_{MN}\nabla_A\phi\nabla^A\phi \\
&+ g_{MN}\frac{e^{a\phi}}{4n!}F_{[n]}^2 - \frac{e^{a\phi}}{2(n-1)!}F_{M\ldots}F_N\cdots = 0.
\end{aligned}
\tag{A.21}
$$

We use the same method to take the trace:

$$
\begin{aligned}
&R - \frac{D}{2}R - \frac{1}{2}\nabla_R\phi\nabla^R\phi + \frac{D}{4}\nabla_A\phi\nabla^A\phi + \frac{De^{a\phi}}{4n!}F_{[n]}^2 - \frac{e^{a\phi}}{2(n-1)!}F_{[n]}^2 = 0 \\
&\Rightarrow R - \frac{1}{2}\nabla_A\phi\nabla^A\phi - \frac{e^{a\phi}}{2(n-1)!}F^2\frac{\left(1-\frac{D}{2n}\right)}{\left(1-\frac{D}{2}\right)} = 0
\end{aligned}
\tag{A.22}
$$

Putting (A.22) back into (A.21), we obtain the following:

$$
\begin{aligned}
&R_{MN} - \frac{1}{2}\nabla_M\phi\nabla_N\phi - \frac{e^{a\phi}}{2(n-1)!}F_{M\ldots}F_N\cdots + F^2 g_{MN}e^{a\phi}\left(\frac{1}{4n!} - \frac{1}{4(n-1)!}\left(\frac{1-\frac{D}{2n}}{1-\frac{D}{2}}\right)\right) = 0 \\
\Rightarrow &R_{MN} - \frac{1}{2}\nabla_M\phi\nabla_N\phi - \frac{e^{a\phi}}{2(n-1)!}F_M\ldots F_N\cdots + F^2 g_{MN}e^{a\phi}\frac{1}{4n!}\frac{1-n}{1-\frac{D}{2}} = 0 \\
\Rightarrow &R_{MN} = \frac{1}{2}\nabla_M\phi\nabla_N\phi + \frac{e^{a\phi}}{2(n-1)!}\left(F_{M\ldots}F_N\cdots - \frac{n-1}{n(D-2)}F^2 g_{MN}\right) = 0.
\end{aligned}
\tag{A.23}
$$

Here, we prove the first two equations in (3.10), showing that

$$
S_{MN} = \frac{e^{a\phi}}{2(n-1)!}\left(F_{M\ldots}F_N\cdots - \frac{n-1}{n(D-2)}F^2 g_{MN}\right).
\tag{A.24}
$$

A.2.2 Variation with Respect to the Gauge Potential

Only the field strength contains the gauge potential, which enables us to focus on the last term. We have performed this calculation in Section A.1. Using (A.10), we find that

$$
\begin{aligned}
&\delta\int dx^D\frac{Fg}{2n!}e^{a\phi}F_{[n]}^2 \propto \int e^{a\phi}\delta\left(F_{[n]}\wedge *F_{[n]}\right) \\
\Rightarrow &\int dx^D e^{a\phi}\left(\delta\left(A_{[n-1]}\right)\wedge d*F_{[n]}\right),
\end{aligned}
\tag{A.25}
$$

resulting in

$$
e^{a\phi}d*F_{[n]} = 0.
\tag{A.26}
$$

Applyinh the Hodge star to both sides yields the following:

$$\begin{aligned} &*e^{a\phi}d*F_{[n]}=0 \\ &\Rightarrow \nabla\left(e^{a\phi}F_{[n]}\right)=0. \end{aligned} \tag{A.27}$$

A.2.3 Variation with Respect to ϕ

The second and last terms contain ϕ. We first operate on the second term:

$$\begin{aligned} &\int dx^D\delta\left(g^{RS}\nabla_R\phi\nabla_S\phi\right) \\ =&\int dx^D\left(g^{RS}\left(\nabla_R\delta\phi\right)\nabla_S\phi+g^{RS}\nabla_R\phi\left(\nabla_S\delta\phi\right)\right) \\ =&\int dx^D\left(2g^{RS}\nabla_R\phi\nabla_S\delta\phi\right) \\ =&2\int dx^Dg^{RS}\nabla_S\left(\nabla_R\phi\delta\phi\right)-2\int dx^Dg^{RS}\nabla_S\nabla_R\phi\delta\phi \\ =&-2\int dx^D\nabla^R\nabla_R\phi\delta\phi=-2\int dx^D\Box\phi\delta\phi. \end{aligned} \tag{A.28}$$

For the last term, we have

$$\int dx^D\delta\left(e^{a\phi}F_{[n]}^2\right)=\int dx^Dae^{a\phi}F_{[n]}^2\delta\phi. \tag{A.29}$$

Combining the above equations leads to

$$\Box\phi=\frac{a}{2n!}e^{a\phi}F_{[n]}^2. \tag{A.30}$$

A.3 Derivation of 1-Form Spin Connection

Because the torsion vanishes, we obtain the 1-form spin connection

$$\mathrm{d}e^{\underline{E}}+\omega^{\underline{E}}{}_{\underline{F}}\wedge e^{\underline{F}}=0. \tag{A.31}$$

We can find the value of ω by setting $\underline{E}=\underline{\mu}$:

$$\begin{aligned} &de^{\underline{\mu}}+\omega^{\underline{\mu}}{}_{\underline{E}}\wedge e^{\underline{E}}=0 \\ \Rightarrow &d\left(e^{A(r)}dx^\mu\right)+\omega^{\underline{\mu}}{}_{\underline{v}}\wedge e^{\underline{v}}+\omega^{\underline{\mu}}{}_{\underline{m}}\wedge e^{\underline{m}}=0 \\ \Rightarrow &d\left(e^{A(r)}dx^\mu\right)+\omega^{\underline{\mu}}{}_{\underline{v}}\wedge e^{A(r)}dx^\nu+\omega^{\underline{\mu}}{}_{\underline{m}}\wedge e^{B(r)}dy^m=0 \\ \Rightarrow &d\left(e^{A(r)}\right)dx^\mu+d\left(dx^\mu\right)e^{A(r)}+\omega^{\underline{\mu}}{}_{\underline{v}}\wedge e^{A(r)}dx^\nu+\omega^{\underline{\mu}}{}_{\underline{m}}\wedge e^{B(r)}dy^m=0 \\ \Rightarrow &e^{A(r)}A'(r)dr\wedge dx^\mu+\omega^{\underline{\mu}}{}_{\underline{v}}\wedge e^{A(r)}dx^\nu+\omega^{\underline{\mu}}{}_{\underline{m}}\wedge e^{B(r)}dy^m=0. \end{aligned} \tag{A.32}$$

The above equation has three terms. Recalling that $r=\sqrt{y^my^m}$, we find that the first and last terms both contain transverse elements, which ensures that the middle term must vanish as follows:

$$\begin{aligned} &\omega^{\underline{\mu}}{}_{\underline{v}}\wedge e^{A(r)}dx^v=0 \\ \Rightarrow\quad &\omega^{\underline{\mu}}{}_{\underline{v}}=0 \\ \Rightarrow\quad &\omega^{\underline{\mu}\underline{v}}=0. \end{aligned} \tag{A.33}$$

Substituting $dr = \frac{y^m}{r}dy^m$ into (A.32) leads to

$$\begin{aligned} & e^{A(r)}A'(r)\frac{y^n}{r}dy^n \wedge dx^\mu + \omega^{\underline{\mu}}{}_{\underline{m}} \wedge e^{B(r)}dy^m = 0, \\ \Rightarrow & -e^{A(r)}A'(r)\frac{y^n}{r}dx^\mu \wedge dy^n + \omega^{\underline{\mu}}{}_{\underline{m}} \wedge e^{B(r)}dy^m = 0, \\ \Rightarrow & (-e^{A(r)}A'(r)\frac{y^m}{r}dx^\mu + \omega^{\underline{\mu}}{}_{\underline{m}}e^{B(r)}) \wedge dy^m = 0, \\ \Rightarrow & \omega^{\underline{\mu}}{}_{\underline{m}}e^{B(r)} = e^{A(r)}A'(r)\frac{y^m}{r}dx^\mu, \\ \Rightarrow & \omega^{\underline{\mu}}{}_{\underline{m}} = e^{-B(r)}e^{A(r)}\frac{\partial y^m}{\partial r}\frac{\partial}{\partial y^m}A(r)\frac{y^m}{r}dx^\mu, \\ \Rightarrow & \omega^{\underline{\mu}}{}_{\underline{m}} = e^{-B(r)}\partial_m A(r)e^{\underline{\mu}}, \\ \Rightarrow & \omega^{\underline{\mu}\underline{m}} = e^{-B(r)}\partial_m A(r)e^{\underline{\mu}}. \end{aligned} \tag{A.34}$$

We then perform a similar process by taking $\underline{E} = \underline{m}$:

$$\begin{aligned} & de^{\underline{m}} + \omega^{\underline{m}}{}_{\underline{E}} \wedge e^{\underline{E}} = 0, \\ \Rightarrow & d(e^{B(r)}dx^m) + \omega^{\underline{m}}{}_{\underline{v}} \wedge e^{\underline{v}} + \omega^{\underline{m}}{}_{\underline{n}} \wedge e^{\underline{n}} = 0, \\ \Rightarrow & e^{B(r)}\partial_n B(r)dy^n \wedge dy^m + \omega^{\underline{m}}{}_{\underline{n}} \wedge e^{B(r)}dy^n = 0, \\ \Rightarrow & \omega^{\underline{m}}{}_{\underline{n}} = \partial_n B(r)dy^m, \\ \Rightarrow & \omega^{\underline{m}}{}_{\underline{n}} = e^{-B(r)}\partial_n B(r)e^{\underline{m}}. \end{aligned} \tag{A.35}$$

From this, we obtain the following relation:

$$\omega^{\underline{mn}} = e^{-B(r)}\partial_n B(r)e^{\underline{m}} - e^{-B(r)}\partial_m B(r)e^{\underline{n}}. \tag{A.36}$$

A.4 Derivation of Curvature via Vielbein

Now, we need to relate the 2-form connection to the general Ricci tensor. We can construct a 2-form connection as $R^{\underline{EF}} = \frac{1}{2}R^{\underline{EF}}{}_{\underline{IJ}}e^{\underline{I}} \wedge e^{\underline{J}}$, which can be combined with

$$R_{MN} = R_{\underline{MN}}e^{\underline{M}}{}_{M}e^{\underline{N}}{}_{N}. \tag{A.37}$$

A.4.1 Ricci Tensor in Worldvolume

Let us start with the Ricci tensor in the worldvolume, which can be formed by tracing over both the worldvolume and transverse space on the Riemann tensor:

$$R_{\underline{\nu\lambda}} = R^{\underline{\mu}}{}_{\underline{\nu\mu\lambda}} + R^{\underline{m}}{}_{\underline{\nu m\lambda}}. \tag{A.38}$$

The first part can be derived via

$$R^{\underline{\mu\nu}} = \frac{1}{2}R^{\underline{\mu\nu}}{}_{\underline{\rho\sigma}}\hat{e}^{\underline{\rho}} \wedge \hat{e}^{\underline{\sigma}} = -\partial_m A\partial_m A e^{-2B}\delta^{\underline{\mu}}{}_{\underline{\rho}}\delta^{\underline{\nu}}{}_{\underline{\sigma}}\hat{e}^{\underline{\rho}} \wedge \hat{e}^{\underline{\sigma}}. \tag{A.39}$$

The antisymmetric property of the wedge product can be used, leading to

$$\begin{aligned} R^{\underline{\mu\nu}}{}_{\underline{\rho\sigma}} = & -\partial_m A\partial_m A e^{-2B}(\delta^{\underline{\mu}}{}_{\underline{\rho}}\delta^{\underline{\nu}}{}_{\underline{\sigma}} - \delta^{\underline{\mu}}{}_{\underline{\sigma}}\delta^{\underline{\nu}}{}_{\underline{\rho}}), \\ \Rightarrow R^{\underline{\mu}}{}_{\underline{\nu\mu\lambda}} = & -\partial_m A\partial_m A e^{-2B}(\delta^{\underline{\mu}}{}_{\underline{\mu}}\eta_{\underline{\nu\sigma}} - \eta_{\underline{\nu\sigma}}), \\ \Rightarrow R^{\underline{\mu}}{}_{\underline{\nu\mu\lambda}} = & -(d-1)\eta_{\underline{\nu\sigma}}\partial_m A\partial_m A e^{-2B}. \end{aligned} \tag{A.40}$$

Then, we perform a similar process:

$$
\begin{aligned}
R^{\underline{\mu m}} &= \frac{1}{2} R^{\underline{\mu m}}{}_{\underline{AB}} \hat{e}^{\underline{A}} \wedge \hat{e}^{\underline{B}}, \\
&= \frac{1}{2} (R^{\underline{\mu m}}{}_{\underline{k\sigma}} \hat{e}^{\underline{k}} \wedge \hat{e}^{\underline{\sigma}} + R^{\underline{\mu m}}{}_{\underline{\sigma k}} \hat{e}^{\underline{\sigma}} \wedge \hat{e}^{\underline{k}}), \\
&= R^{\underline{\mu m}}{}_{\underline{k\sigma}} \hat{e}^{\underline{k}} \wedge \hat{e}^{\underline{\sigma}}.
\end{aligned}
\tag{A.41}
$$

Cancelling the vielbein basis and taking the trace yields the following:

$$
R^{\underline{m}}{}_{\underline{\nu m \sigma}} = -e^{-2B} \eta_{\underline{\nu\sigma}} (\partial_n^2 A + (\partial_n A)^2 + (D-d-2)\partial_n A \partial_n B). \tag{A.42}
$$

Combining the two parts and applying the spherical symmetry leads to

$$
R_{\mu\nu} = -\eta_{\mu\nu} e^{2(A-B)} \left(A'' + dA'^2 + \tilde{d} A' B' + \frac{\tilde{d}+1}{r} A' \right), \tag{A.43}
$$

where we note that $\tilde{d} = D - d - 2$.

A.4.2 Ricci Tensor in Transverse Space

We reproduce the above method in the transverse space as follows:

$$
R_{\underline{nm}} = R^{\underline{\mu}}{}_{\underline{n\mu m}} + R^{\underline{k}}{}_{\underline{nkm}}. \tag{A.44}
$$

Using the antisymmetric property of the wedge product again, we obtain

$$
\begin{aligned}
R^{\underline{nm}} &= \frac{1}{2} R^{\underline{nm}}{}_{\underline{ab}} \hat{e}^{\underline{a}} \wedge \hat{e}^{\underline{b}}, \\
\Rightarrow R^{\underline{nm}}{}_{\underline{ab}} &= e^{-2B} ((\partial_k \partial_m B - \partial_k \partial_m B)(\delta^{\underline{k}}{}_{\underline{a}} \delta^{\underline{n}}{}_{\underline{b}} - \delta^{\underline{k}}{}_{\underline{b}} \delta^{\underline{n}}{}_{\underline{a}}) \\
&\quad - (\partial_k \partial_n B - \partial_n \partial_k B)(\delta^{\underline{k}}{}_{\underline{a}} \delta^{\underline{m}}{}_{\underline{b}} - \delta^{\underline{k}}{}_{\underline{b}} \delta^{\underline{m}}{}_{\underline{a}}) \\
&\quad - \partial_k B \partial_k B (\delta^{\underline{n}}{}_{\underline{a}} \delta^{\underline{m}}{}_{\underline{b}} - \delta^{\underline{n}}{}_{\underline{b}} \delta^{\underline{m}}{}_{\underline{a}})), \\
\Rightarrow R^{\underline{n}}{}_{\underline{pnb}} &= e^{-2B} (-(\partial_b \partial_p B - \partial_b \partial_p B)\tilde{d} - \delta_{\underline{bp}} (\partial_n B)^2 \tilde{d} - \delta_{\underline{bp}} \partial_n^2 B).
\end{aligned}
\tag{A.45}
$$

The part $R^{\underline{\mu}}{}_{\underline{n\mu m}}$ has been derived in the previous subsection, and we only need to exchange its indices:

$$
\begin{aligned}
R^{\underline{\nu m}}{}_{\underline{\sigma b}} &= e^{-2B} ((\partial_n \partial_m A + \partial_m A \partial_n A - \partial_m A \partial_n B - \partial_n A \partial_m B) \delta^{\underline{n}}{}_{\underline{b}} \delta^{\underline{\nu}}{}_{\underline{\sigma}} + \partial_n A \partial_n B \delta^{\underline{m}}{}_{\underline{b}} \delta^{\underline{\nu}}{}_{\underline{\sigma}}), \\
\Rightarrow R^{\underline{\nu}}{}_{\underline{p\nu b}} &= -e^{-2B} ((\partial_b \partial_p A + \partial_p A \partial_b A - \partial_p A \partial_b B - \partial_b A \partial_p B) d + \delta_{\underline{bp}} \partial_n A \partial_n B d).
\end{aligned}
\tag{A.46}
$$

Summarising the above calculation, we obtain the following:

$$
\begin{aligned}
R_{mn} = &- \delta_{mn} \left(B'' + dA'B' + \tilde{d} B'^2 + \frac{2\tilde{d}+1}{r} B' + \frac{d}{r} A' \right) \\
&- \frac{y^m y^n}{r^2} \left(\tilde{d} B'' + dA'' - 2dA'B' + dA'^2 - \tilde{d} B'^2 - \frac{\tilde{d}}{r} B' - \frac{d}{r} A' \right).
\end{aligned}
\tag{A.47}
$$

A.5 Derivation of Magnetic Field Strength

We can deduce the power of r via Biachi identity:

$$\partial_q F_{m_1\cdots m_n} = r^{-(n+1)} \left(\epsilon_{m_1\cdots m_n q} - (n+1)\epsilon_{m_1\cdots m_n p} y^p y_q / r^2\right), \tag{A.48}$$

where the maximum form can be reached is $(n+1)$ form. In that case, any $(n+2)$ form would vanish:

$$\begin{aligned} 0 =& \epsilon_{[m_1\cdots m_n p} y_{q]}. \\ \Rightarrow 0 =& \epsilon_{[m_1\cdots m_n q} y_{p]}. \\ \Rightarrow 0 =& \frac{1}{(n+2)!} \left(\epsilon_{m_1\cdots m_n q} y_p + (-1)^{n+1} \epsilon_{p m_1\cdots m_n} y_q (n+1) + \cdots\right). \end{aligned} \tag{A.49}$$

Here we antisymmetrized a $(n+2)$ form and found similar terms in (A.48). Multiply y^p on both sides of (A.49), We got:

$$\begin{aligned} &(\epsilon_{m_1\cdots m_n q} y_p)\, y^p + (n+1)(-1)^{n+1}(-1)^n \epsilon_{m_1\cdots m_n p} y_q y_p = 0, \\ &\Rightarrow (\epsilon_{m_1\cdots m_n q}) = (n+1)\epsilon_{m_1\cdots m_n p} y_q y_p r^{-2}, \end{aligned} \tag{A.50}$$

leading to Bianchi identity requirement for $F_{[n]}$. Again, it shows our ansatz fulfils the conditions.

A.6 Derivation of Brane Orbit

Continuing with our relabel on einbeins, one can express our time derivative component in terms of L:

$$\begin{aligned} L^2 &= e^{6A}\dot{t}^2 - e^{3A}\dot{y}^m\dot{y}^m, \\ \Rightarrow \dot{t} &= \left(e^{-6A}L^2 + e^{-3A}\dot{y}^m\dot{y}^m\right)^{\frac{1}{2}}, \end{aligned} \tag{A.51}$$

which can further substitute this back to the expression for ε:

$$\begin{aligned} \varepsilon &= \frac{m}{2L}\left[2e^{6A}\left(e^{-6A}L^2 + e^{-3A}y^m\dot{y}^m\right)^{\frac{1}{2}}\right] + me^{3A}, \\ \Rightarrow \varepsilon &= m\left[e^{3A}\left(1 + \frac{e^{3A}\dot{y}^m\dot{y}^m}{L^2}\right)^{\frac{1}{2}}\right] + me^{3A}. \end{aligned} \tag{A.52}$$

Now, our energy term depends on $(r, \dot{r}, \dot{\phi})$. In order to get a equation purely depend on radius and its derivative, we can then replace $\dot{\phi}$ via angualr:

$$\begin{aligned} J &= \frac{m}{L}e^{3A}r^2\dot{\phi}, \\ \Rightarrow \dot{\phi} &= J\frac{L}{m}r^{-2}e^{-3A}, \end{aligned} \tag{A.53}$$

which can be put into the istropic $\dot{y}^m\dot{y}^m$ term. Thus, (A.52) only depends on radius and its velocity. To get the equation for radius (3.46), we need to rearrange the energy term:

$$\begin{aligned} &\varepsilon^2 e^{-6A} - 2\varepsilon m e^{-3A} + m^2 = m^2 + e^{3A}\left(\dot{r}^2 + r^2\dot{\phi}^2\right)\frac{m^2}{L^2}, \\ \Rightarrow &\varepsilon^2 e^{-6A} - 2\varepsilon m e^{-3A} = \frac{m^2}{L^2}e^{3A}\dot{r}^2 + J^2 r^{-2} e^{-3A}, \\ \Rightarrow &\dot{r}^2 = \frac{L^2}{m^2}e^{-3A}\left(\varepsilon^2 e^{-6A} - 2\varepsilon e^{-3A} - \frac{J^2}{r^2}e^{-3A}\right). \end{aligned} \tag{A.54}$$

Appendix B

Dimensional Reduction Calculations

B.1 Chern–Simons Dimensional Reduction

Here we implicate the reduction rule the Chern–Simons term [33].

$$
\begin{aligned}
D = 10 :& \frac{1}{2}\tilde{F}_4 \wedge \tilde{F}_4 \wedge A_2, \\
D = 9 :& \left(-\frac{1}{4}\tilde{F}_4 \wedge \tilde{F}_4 \wedge A_1^{ij} - \frac{1}{2}\tilde{F}_3^i \wedge \tilde{F}_3^j \wedge A_3 \right) \epsilon_{ij}, \\
D = 8 :& \left(-\frac{1}{12}\tilde{F}_4 \wedge \tilde{F}_4 A_0^{ijk} - \frac{1}{6}\tilde{F}_3^i \wedge \tilde{F}_3^j \wedge A_2^k + \frac{1}{2}\tilde{F}_3^i \wedge \tilde{F}_2^{jk} \wedge A_3 \right) \epsilon_{ijk}, \\
D = 7 :& \left(-\frac{1}{6}\tilde{F}_4 \wedge \tilde{F}_3^i A_0^{jkl} + \frac{1}{6}\tilde{F}_3^i \wedge \tilde{F}_3^j \wedge A_1^{kl} + \frac{1}{8}\tilde{F}_2^{ij} \wedge \tilde{F}_2^{kl} \wedge A_3 \right) \epsilon_{ijkl}, \\
D = 6 :& \left(\frac{1}{12}\tilde{F}_4 \wedge \tilde{F}_2^{ij} A_0^{klm} + \frac{1}{12}\tilde{F}_3^i \wedge \tilde{F}_3^j A_0^{klm} + \frac{1}{8}\tilde{F}_2^{ij} \wedge \tilde{F}_2^{kl} \wedge A_2^m \right) \epsilon_{ijklm}, \\
D = 5 :& \left(\frac{1}{12}\tilde{F}_3^i \wedge \tilde{F}_2^{jk} A_0^{lmn} - \frac{1}{48}\tilde{F}_2^{ij} \wedge \tilde{F}_2^{kl} \wedge A_1^{mn} - \frac{1}{72}\tilde{F}_1^{ijk} \wedge \tilde{F}_1^{lmn} \wedge A_3 \right) \epsilon_{ijklmn}, \\
D = 4 :& \left(-\frac{1}{48}\tilde{F}_2^{ij} \wedge \tilde{F}_2^{kl} A_0^{mnp} - \frac{1}{72}\tilde{F}_1^{ijk} \wedge \tilde{F}_1^{lmn} \wedge A_2^p \right) \epsilon_{ijklmnp}, \\
D = 3 :& \frac{1}{144}\tilde{F}_1^{ijk} \wedge \tilde{F}_1^{lmn} \wedge A_1^{pq} \epsilon_{ijklmnpq}, \\
D = 2 :& \frac{1}{1296}\tilde{F}_1^{ijk} \wedge \tilde{F}_1^{lmn} A_0^{pqr} \epsilon_{ijklmnpqr}.
\end{aligned}
\tag{B.1}
$$

B.2 Local and Global Symmetries in S^1 Compactification

In this section, we will consider some important calculations that influence local and global symmetries. Recall that we have the local transformation relation (4.33). By choosing (M, N) as (z, z), (μ, z) and (μ, ν), we can start with G_{zz} as follows:

$$
\begin{aligned}
\delta G_{zz} &= \xi^\rho 2\beta G_{zz} \partial_\rho \phi, \\
\delta \left(e^{2\beta\phi} \right) &= 2\beta G_{zz} \delta\phi, \\
\Rightarrow \delta\phi &= \xi^\rho \partial_\rho \phi,
\end{aligned}
\tag{B.2}
$$

where we have substituted the ansatz for G_{zz} in the second equation. A similar process can be used for the rest of the calculations. For $G_{\mu z}$, we find that

$$\begin{aligned}
\delta G_{\mu z} &= \xi^\rho \partial_\rho G_{\mu z} + G_{\rho z}\partial_\mu \xi^\rho + G_{zz}\partial_\mu \xi^z, \\
&= \xi^\rho e^{2\beta\phi}\partial_\rho \mathcal{A}_\mu + 2\beta G_{\mu\nu}\phi\phi + e^{2\beta\phi}\mathcal{A}_\rho \partial_\mu \xi^\rho + e^{2\beta\phi}\partial_\mu \lambda, \\
\delta\left(e^{2\beta\phi}\mathcal{A}_\mu\right) &= 2\beta G_{\mu z}\delta\phi + e^{2\beta\phi}\delta\mathcal{A}_\mu, \\
\Rightarrow \delta\mathcal{A}_\mu &= \xi^\rho \partial_\rho \mathcal{A}_\mu + \mathcal{A}_\rho \partial_\mu \xi^\rho + \partial_\mu \lambda.
\end{aligned} \tag{B.3}$$

For $G_{\mu\nu}$, we have

$$\begin{aligned}
&\delta G_{\mu\nu} = \xi^\rho \partial_\rho G_{\mu\nu} + G_{\rho\nu}\partial_\mu \xi^\rho + G_{\mu\rho}\partial_\nu \xi^\rho + G_{z\nu}\partial_\mu \xi^z + G_{\mu z}\partial_\nu \xi^z, \\
&\delta\left(e^{2\alpha\phi}g_{\mu\nu} + e^{2\beta\phi}\mathcal{A}_\mu\mathcal{A}_\nu\right) = 2\alpha e^{2\alpha\phi}g_{\mu\nu}\delta\phi + e^{2\alpha\phi}\delta g_{\mu\nu} \\
&+ 2\beta e^{2\beta\phi}\mathcal{A}_\mu\mathcal{A}_\nu\delta\phi + e^{2\beta\phi}\delta\mathcal{A}_\mu\mathcal{A}_\nu + e^{2\beta\phi}\mathcal{A}_\mu\delta\mathcal{A}_\nu,
\end{aligned} \tag{B.4}$$

which leads to

$$\delta g_{\mu\nu} = \xi^\rho \partial_\rho g_{\mu\nu} + g_{\rho\nu}\partial_\mu \xi^\rho + g_{\mu\rho}\partial_\nu \xi^\rho, \tag{B.5}$$

which proves the local symmetry of the D-dimensional metric.

Now, let us consider the global symmetry calculations. In this case, we ignore the x dependence on our infinitesimal transformations, which gives the general expression for a metric in $(D + 1)$ dimensions in (4.38). For G_{zz}, we can derive the following equation:

$$\begin{aligned}
\delta G_{zz} &= 2cG_{zz} + 2aG_{zz}, \\
\delta\left(e^{2\beta\phi}\right) &= 2\beta\delta\phi G_{zz}, \\
\Rightarrow \beta\delta\phi &= c + a.
\end{aligned} \tag{B.6}$$

We can use the above result to further calculate $G_{z\mu}$ as follows:

$$\begin{aligned}
\delta G_{z\mu} &= cG_{z\mu} + 2aGz\mu, \\
\delta\left(e^{2\beta\phi}\mathcal{A}_\mu\right) &= e^{2\beta\phi}\delta\mathcal{A}_\mu + 2\beta G_{z\mu}\delta\phi, \\
\Rightarrow e^{2\beta\phi}\delta\mathcal{A}_\mu &= -cG_{z\mu}, \\
\Rightarrow \delta\mathcal{A}_\mu &= -c\mathcal{A}_\mu.
\end{aligned} \tag{B.7}$$

Using both expressions for $\delta\phi$ and $\delta\mathcal{A}_\mu$, we can perform the final transformation:

$$\begin{aligned}
\delta G_{\mu\nu} &= 2ae^{2\alpha\phi}g_{\mu\nu} + 2ae^{2\beta\phi}\mathcal{A}_\mu\mathcal{A}_\nu, \\
\Rightarrow \delta g_{\mu\nu} &= 2ag_{\mu\nu} - 2\alpha g_{\mu\nu}\delta\phi,
\end{aligned} \tag{B.8}$$

which shows global symmetry in the dimensional reduction.